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Riemann Surfaces

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Exercises for Chapter 9: Line bundles and divisors

Exercise 1. Tensor product of line bundles

(1) Let us call *(complex) line* a 1-dimensional complex vector space. Show that for any complex lines L and L':

$$\begin{aligned} L^* &\text{ is a line} \\ L \otimes L' &\text{ is a line} \\ L \otimes L' &\approx L' \otimes L \\ L \otimes \mathbb{C} &\approx \mathbb{C} \otimes L \approx L \\ L \otimes L^* &\approx L^* \otimes L \approx \mathbb{C} \end{aligned}$$

All the isomorphisms above must be *canonical*: they do not depend on the choice of a basis. *You may use the following definition for the tensor product of finite-dimensional vector spaces:* $V \otimes W = \mathrm{Hom}(V^*, W)$.

(2) Let X be a Riemann surface. Show that the set of holomorphic line bundles over X is a group for the tensor product. (Show that the group structure descends to the set of isomorphism classes of line bundles).

Exercise 2. Line bundles vs sheaves

Let X be a Riemann surface. Write a complete and detailed proof for the bijective correspondence

$$\{\text{isomorphism classes of line bundles on } X\} \quad \leftrightarrow \quad \check{H}^1(X, \mathcal{O}_X^*) \ .$$

Show moreover that it is a group isomorphism for the appropriate group structure on both sides.

Exercise 3. Néron-Severi group of a Riemann surface

Let X be a compact Riemann surface. Show that

$$\mathrm{Pic}(X) \big/ \mathrm{Pic}_0(X) \approx \mathrm{Div}(X) \big/ \mathrm{Div}_0(X) \approx \mathbb{Z} \ .$$

Exercise 4. Principal divisors vs degree zero divisors, part 1

Let X be a compact Riemann surface. The goal of this exercise is to show that any principal divisor has zero degree.

(1) Explain why the goal of the exercise amounts to showing that any meromorphic function f on a compact Riemann surface has as many zeros as it has poles, counted with multiplicity.

(2) Prove the following *Argument principle* for open sets of $\mathbb{C}$. Let f be a meromorphic function on a simply connected open set $U \subseteq \mathbb{C}$ and let γ be a positively oriented simple closed curve in U not going through any zero or pole of f. Denote K the compact set bounded by γ. Then

$$\frac{1}{2i\pi} \int_K \frac{f'(z)}{f(z)} \, \mathrm{d}z = Z_K(f) - P_K(f)$$

where $Z_K(f)$ (resp. $P_K(f)$) is the number of zeros (resp. poles) of f in K, counted with multiplicity. *Hint: use the residue theorem.*

(3) Show that the *residue at a pole* of a meromorphic 1-form on a Riemann surface is well-defined. On the contrary, illustrate with an example that the residue at a pole of a meromorphic function is *not* well-defined. *Note that this is not so surprising: on a Riemann surface, taking the integral of a function along a curve does not make sense, whereas it does for a* 1*-form. So the residue theorem is really about* 1*-forms, not functions.*

(4) You may admit or prove the following *Residue theorem for compact Riemann surfaces*. Let ω be a meromorphic 1-form on X (we can also allow isolated essential singularities, the proof stays the same). Then

$$\sum_{p \in X} \mathrm{Res}_p(\omega) = 0 \, .$$

Hint: Follow the proof sketched here: `https://bit.ly/2SyVyfF`.

(5) Conclude by considering the meromorphic 1-form $\omega = \frac{\mathrm{d}f}{f}$.

Exercise 5. Principal divisors vs degree zero divisors, part 2: the case of $\mathbb{C}P^1$

The goal of this exercise is to show that the converse of Exercise 4 is true when $X = \mathbb{C}P^1$.

(1) Consider a divisor $D = \sum c_k [a_k : b_k]$ (we use homogeneous coordinates on $\mathbb{C}P^1$). Assuming D has degree zero, show that the function

$$f([z : w]) = \prod_k (b_k z - a_k w)^{c_k}$$

is a well-defined meromorphic function on X with $(f) = D$. Conclude.

(2) Show that $\mathrm{Pic}(\mathbb{C}P^1) \approx \mathbb{Z}$.

Exercise 6. Picard group of $\mathbb{C}P^1$

In this exercise, we classify the line bundles over $X = \mathbb{C}P^1$.

(1) Using Exercise 5, show that $\deg\colon \operatorname{Div}(X) \to \mathbb{Z}$ induces a group isomorphism (still abusively denoted deg)
$$\deg\colon \operatorname{Pic}(\mathbb{C}P^1) \xrightarrow{\sim} \mathbb{Z}\,.$$

(2) Let us denote by L the tautological line bundle on $\mathbb{C}P^1$. We recall that by definition, this line bundle is given by the projection $\mathbb{C}^2 \to \mathbb{C}^2/\sim$ with $\mathbb{C}^2/\sim = \mathbb{C}\mathrm{P}^1$. In particular, the line L_p above a point $p \in \mathbb{C}P^1$ is p itself, seen as a line in $\mathbb{C}^2$. Prove carefully that L is indeed a holomorphic line bundle.

(3) Find a meromorphic section of L. What is the degree of L?

(4) Let us denote $\mathcal{O}(1)$ the dual line bundle of L. Show that $\deg(\mathcal{O}(1)) = 1$.

(5) Let us denote $\mathcal{O}(k)$ the line bundle obtained by taking k tensor products of $\mathcal{O}(1)$. For the negative values of $k \in \mathbb{Z}$, we define $\mathcal{O}(k)$ as the dual line bundle of $\mathcal{O}(-k)$. For $k = 0$, we define $\mathcal{O}(k)$ as the trivial line bundle. What is the degree of $\mathcal{O}(k)$?

(6) Show that $k \mapsto \mathcal{O}(k)$ defines an isomorphism from $\mathbb{Z}$ to $\operatorname{Pic}(\mathbb{C}P^1)$ and that it is the inverse of $\deg\colon \operatorname{Pic}(\mathbb{C}P^1) \xrightarrow{\sim} \mathbb{Z}$.

(7) Find a meromorphic 1-form on $\mathbb{C}\mathrm{P}^1$ and derive the degree of the canonical line bundle K. Show that $K \approx \mathcal{O}(-2)$.

Exercise 7. Principal divisors vs degree zero divisors, part 3

Let X be a compact Riemann surface. The goal of this exercise is to show that the converse of Exercise 4 is false unless $X = \mathbb{C}P^1$ (cf Exercise 5 for the case $X = \mathbb{C}P^1$).

(1) First we need to know that any nonconstant holomorphic map between compact Riemann surfaces $f\colon X \to Y$ is a *branched covering*. (You may admit the answers to following questions if you want to quickly finish the exercise.)
   (a) Show that for every $p \in X$, one can find a local coordinate z at p and a local coordinate w at Y such that the function f is written $w = z^k$. The integer $k \in \mathbb{N}$ is called the *order of f at p.* and denoted $\operatorname{ord}_p(f)$.
   (b) Show that the map
$$\begin{aligned} Y &\to \mathbb{N} \\ q &\mapsto \sum_{p \in f^{-1}(y)} \operatorname{ord}_p(f) \end{aligned}$$
   is locally constant. *Where did you use the assumption that X is compact? You really need it, and not just to say that the sum is finite.* Hence, it is constant (Y is connected). Its value is called the *degree* of f.

(c) Show that if f has degree 1 then it is a covering map. *You may admit the following topological lemma: any local homeomorphism between compact, Hausdorff, connected topological spaces is a covering map.*

(2) Consider the divisor $D = (p) - (p')$ on a compact Riemann surface X. Assume that there exists a meromorphic function f such that $D = (f)$. Show that f is a holomorphic map from X to the Riemann sphere $Y = \hat{\mathbb{C}}$ of degree 1. Conclude.

Exercise 8. Bounds on $\dim H^0(X, D)$ **in terms of** $\deg D$**.**

Let X be a compact Riemann surface.

(1) Show that if $\deg D < 0$, then $\dim H^0(X, D) = 0$.

(2) Show that for any $p \in X$, $\dim H^0(X, D + p) \leqslant \dim H^0(X, D)$. *Hint: Call $m \in \mathbb{Z}$ the order of D at p. Let $f \in H^0(X, D + p)$. Write the Laurent expansion of f in a local chart at p: $f(z) = a_{-m-1}z^{-m-1} + a_{-m}z^{-m} + \dots$. Argue that $f \mapsto a_{-m-1}$ is a linear map from $H^0(X, D + p)$ to $\mathbb{C}$ with kernel $H^0(X, D)$ and conclude.*

(3) Show that for any divisor D of nonnegative degree, $\dim H^0(X, D) \leqslant \deg(D)$. *Hint: Write a proof by induction.*

(4) Using the Riemann-Roch theorem, show that

$$\deg(D) + 1 - g \leqslant \dim H^0(X, D) \leqslant \deg(D)$$

The first inequality is called Riemann's inequality.

Exercise 9. Dimension of $H^0(X, K^2)$

Let X be a compact Riemann surface of genus g. Denote by K its canonical bundle; we recall that $\deg(K) = 2g - 2$. Denote by K^2 the line bundle $K \otimes K$ (note that as a divisor, one should write $2K$ rather than K^2). Sections of K^2 are called *holomorphic quadratic differentials*, they look like $\varphi(z)\,\mathrm{d}z^2$ (where φ is a holomorphic function) in a local complex coordinate z. The goal of this exercise is to compute $\dim H^0(X, K^2)$.

(1) Write the Riemann-Roch theorem for the divisor $2K$.

(2) Conclude that $\dim H^0(X, K^2) = 3g - 3$ when $g > 1$.

(3) Show that if $g = 1$, then $\dim H^0(X, -K) = 1$. Conclude that $\dim H^0(X, K^2) = 1$.

(4) Show that any holomorphic vector field on the Riemann sphere is of the form $P(z)\frac{\partial}{\partial z}$ where P is a polynomial of degree $\leqslant 2$. Derive that if $g = 0$, then $\dim H^0(X, -K) = 3$. Conclude that $\dim H^0(X, K^2) = 0$.

(5) One can show that the tangent space at X to the moduli space of Riemann surfaces of genus g is $\mathrm{T}_X\mathcal{M}_g = H^0(X, K^2)$. In particular, the complex dimension of $\mathcal{M}_g$ is $\dim H^0(X, K^2)$. Are the results you found in the two previous questions consistent with what you know about $\mathcal{M}_g$ for $g = 0$ and $g = 1$?