Brice Loustau
TU Darmstadt

Riemann Surfaces

Winter 2018-2019

Exercises for Chapter 8: Sheaves

Exercise 1. Constant sheaf

Let X be a topological space and let A be a fixed abelian group (or any fixed object of another category).

(1) Recall the definition of the constant presheaf associated to A. Show that it is not a sheaf in general.

(2) Recall the definition of the constant sheaf $\underline{A}$. Show that it is a sheaf.

(3) Can you characterize the sheaf $\underline{A}$ in terms of its stalks?

(4) (Optional). Show that the constant sheaf is the sheafification of the constant presheaf.

Exercise 2. Coboundary map in Čheck cohomology

Let X be a topological space and $\mathcal{U}$ an open cover of X. Consider the Čech cohomology theory in this setting.

(1) Recall the definition of the coboundary map δ.

(2) Show that $\delta \circ \delta = 0$ by direct computation. (You may start with the case $\check{C}^0 \to \check{C}^1 \to \check{C}^2$ as a warm up.)

Exercise 3. Čheck cohomology of order 0

(1) Let X be a topological space and $\mathcal{F}$ a sheaf of abelian groups over X. Show that $H^0(X, \mathcal{F}) \approx \mathcal{F}(X)$ (space of global sections).

(2) Let X be a compact Riemann surface and $\mathcal{O}_X$ be the structure sheaf of X. Show that $H^0(X, \mathcal{F}) \approx \mathbb{C}$.

Exercise 4. First cohomology of smooth functions

Let M be a smooth manifold. Denote by C^∞_M the sheaf of real-valued smooth functions on M. The goal of this exercise is to show that $H^1(M, C^\infty_M) = \{0\}$.

(1) Let $\mathcal{U} = (U_i)_{i\in I}$ be a good cover of M (why does this exist?). Show that a Cech 1-cocycle $f \in \check{Z}^1(M, \mathcal{U}, C^\infty_M)$ is given by a collection $f_{ij} \in C^\infty(U_{ij}, \mathbb{R})$ such that:

$$\forall i, j, k \in I \quad f_{ij\,|U_{ijk}} - f_{ik\,|U_{ijk}} + f_{jk\,|U_{ijk}} = 0\,.$$

(2) Let $(\psi_i)_{i\in I}$ be a partition of unity subordinate to the cover $\mathcal{U}$ (recall the definition; why does this exist?). Let $f = (f_{ij})$ be a 1-cocycle and define, for all $i \in I$,

$$g_i = \sum_{k\in I} \psi_k f_{ik}\,.$$

Show that g_i is a well-defined element of $C^\infty(U_i, \mathbb{R})$.

(3) Compute $g_i - g_j$ and conclude.

(4) Show similarly that $H^1(M, \Omega^k) = \{0\}$ for any k.

Exercise 5. Morphisms of sheaves.

Let X be a topological space and let $\alpha\colon \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves over X.

(1) Recall the definition of what it means for α to be an injective or surjective morphism of sheaves.

(2) Show that α is injective if and only if $\alpha(U)\colon \mathcal{F}(U) \to \mathcal{G}(U)$ is injective for every open set U.

(3) Show that it is false in general that α is surjective if and only if $\alpha(U)\colon \mathcal{F}(U) \to \mathcal{G}(U)$ is surjective for every open set U. *Hint: consider* $\exp\colon \mathcal{O} \to \mathcal{O}^*$ *over* $\mathbb{C}$. Find a proper characterization of surjectivity in terms of the maps $\alpha(U)$.

(4) Show that if $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H}$ is an exact sequence of sheaves, then $0 \to \mathcal{F}(U) \to \mathcal{G}(U) \to \mathcal{H}(U)$ is exact for all U.

Exercise 6. Examples of exact sequences of sheaves

Let X be a Riemann surface. Show that the following sequences of sheaves are exact:

(1) $0 \to \underline{\mathbb{C}} \to \mathcal{A}^0 \xrightarrow{d} \mathcal{Z}^1 \to 0.$
(Here we denote by $\mathcal{Z}^1$ the sheaf of closed complex-valued one-forms.)

(2) $0 \to \underline{\mathbb{C}} \to \mathcal{A}^0 \xrightarrow{d} \mathcal{A}^1 \xrightarrow{d} \mathcal{A}^2 \to 0.$

(3) $0 \to \underline{\mathbb{C}} \to \mathcal{O} \xrightarrow{d} \Omega^1 \to 0.$

(4) $0 \to \mathcal{O} \to \mathcal{A}^0 \xrightarrow{\bar\partial} \mathcal{A}^{0,1} \to 0$

(5) $0 \to \Omega \to \mathcal{A}^{1,0} \xrightarrow{\bar\partial} \mathcal{A}^2 \to 0$

Exercise 7. Exact sequences of sheaves

Consider a short exact sequence of sheaves

$$0 \to \mathcal{F} \xrightarrow{\alpha} \mathcal{G} \xrightarrow{\beta} \mathcal{H} \to 0 \ .$$

Recall why, if $H^1(\mathcal{G}) = 0$, then $H^1(\mathcal{F}) \approx H^0(\mathcal{H})/\beta H^0(\mathcal{G})$.
Show that for any smooth manifold M,

$$H^1_{\mathrm{dR}}(M, \mathbb{R}) = H^1(M, \underline{\mathbb{R}})$$

Hint: Use the result of Exercise 4.