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Riemann Surfaces

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Exercises for Chapter 7: Elliptic Riemann surfaces

Exercise 1. Quotients of $\mathbb{C}$

What are all the Riemann surfaces that arise as a quotient of $\mathbb{C}$? In other words, what Riemann surfaces admit $\mathbb{C}$ as a universal cover? You can try to solve this question on your own, or follow the questions below:

(1) What are the automorphisms of $\mathbb{C}$? By the way, can you prove the answer?

(2) Let Γ be a group acting faithfully and holomorphically on $\mathbb{C}$. Explain why Γ can be described as a subgroup of $\mathrm{Aut}(\mathbb{C})$.

(3) Show that if $\Gamma \leqslant \mathrm{Aut}(\mathbb{C})$ contains an element of the form $z \mapsto az + b$ with $a \neq 0$, then the action of Γ is not free. Derive that if Γ acts freely on $\mathbb{C}$, then Γ may be seen as a subgroup of $\mathbb{C}$ acting on $\mathbb{C}$ by translations.

(4) We recall that a continuous action of a (discrete) group Γ on a Hausdorff topological space X is wandering if and only if for every $x \in X$, there exists a neighborhood U of x such that $gU \cap U = \emptyset$ for all but finitely many $g \in G$. Show that for such an action, the orbit of any point is discrete.

(5) Derive from the previous question that if Γ is a subgroup of $\mathbb{C}$ acting on $\mathbb{C}$ by translations and the action is wandering, then Γ is a lattice, in the sense that either:
- $\Gamma = \{0\}$ (trivial group), or
- $\Gamma = \omega\mathbb{Z}$ with $\omega \in \mathbb{C}^*$, or
- $\Gamma = \omega_1\mathbb{Z} + \omega_2\mathbb{Z}$ where ω_1 and ω_2 are two complex numbers that are linearly independent over $\mathbb{R}$. *This is called a (full rank) lattice*.

(6) Conclude that the Riemann surfaces that arise as quotients of $\mathbb{C}$ are: $\mathbb{C}$ itself, annuli (cylinders), and complex tori.

(7) Show that $\exp\colon \mathbb{C} \to \mathbb{C}^*$ is a covering map. How come this didn't appear in the previous answer? By the way, are annuli with finite modulus covered by $\mathbb{C}$?

Exercise 2. Classification theorem

The goal of this exercise is to prove the classification theorem that we saw in section 7.2 in the lectures.

(1) Show that if X is a complex torus, i.e. $X = \mathbb{C}/\Lambda$ where $\Lambda \subseteq \mathbb{C}$ is a lattice, then X admits a natural nowhere vanishing abelian differential, that we denote $\mathrm{d}z$ somewhat abusively.

Now we'll show that if X is a compact Riemann surface and α is a nowhere vanishing abelian differential on X, then there exists a lattice $\Lambda \subseteq \mathbb{C}$ and a biholomorphism $f\colon X \longrightarrow X/\Lambda$ such that $\alpha = \mathrm{d}z$. So, in the next questions we consider such a pair (X, α).

(2) Consider the universal cover $\pi\colon \tilde{X} \longrightarrow X$ and $\tilde{\alpha} = \pi^*\alpha$. Show that there exists a holomorphic function $F\colon \tilde{X} \longrightarrow \mathbb{C}$ such that $\tilde{\alpha} = \mathrm{d}F$. Show that F can be chosen so that $F^*\,\mathrm{d}z = \tilde{\alpha}$. *Hint: consider* $F(z) = \int_{z_0}^{z} \tilde{\alpha}$.

(3) Show that F is a local biholomorphism.

(4) Show that one can find $r > 0$ such that for every $x \in \tilde{X}$, F restricts to biholomorphism from some neighborhood $D_x \subseteq \tilde{X}$ of x to $D(F(x), r) \subseteq \mathbb{C}$. *First show the result with $r = r_x$, and then use compactness of X to argue that r can be chose independently from x.*

(5) *Purely topological question: skip it if you want.* Let X be a topological space, let $Y = (Y, d)$ be a path connected metric space, and let $F\colon X \longrightarrow Y$ be a map satisfying the property that there exists $r > 0$ such that for every $x \in X$, there exists an open subset $D_x \subseteq X$ containing x such that F induces a homeomorphism from D_x to $D(F(x), r)$. Show that F is a covering map. *Hint: let $\Delta_x := D_x \cap F^{-1}(D(F(x), r/2))$. Show that $D(F(x), r/2)$ is the disjoint union of Δ_y for $y \in F^{-1}(x)$.*

(6) Back to the setting of the exercise, show that F is a global biholomorphism from $\tilde{X}$ to $\mathbb{C}$.

(7) Conclude using Exercise 1.

Exercise 3. Arclength of an ellipse

Show that the formula for the arclength of an ellipse in the Euclidean plane is an elliptic integral.

Exercise 4. Projective closure of algebraic curves

The goal of this exercise is to give some details for defining the projective compactification $X^* \subseteq \mathbb{C}P^2$ of an algebraic curve $X \subseteq \mathbb{C}^2$.

(1) Prove that $\mathbb{C}P^2$ is a complex manifold and that the inclusion $\mathbb{C}^2 \longrightarrow \mathbb{C}P^2$ given by an affine patch is a compactification. You may follow these steps:
   (a) One writes $[z_0 : z_1 : z_2]$ for homogeneous coordinates on $\mathbb{C}P^2$. Consider the set $U_0 := \{z_0 \neq 0\} \subseteq \mathbb{C}P^2$ and define $\varphi_0([z_0 : z_1 : z_2]) = \left(\frac{z_1}{z_0}, \frac{z_2}{z_0}\right)$. Define (U_1, φ_1) and (U_2, φ_2) similarly. Show that this defines a complex atlas.
   (b) By definition, an *affine patch* is $\varphi^{-1}\colon \mathbb{C}^2 \longrightarrow \mathbb{C}P^2$, where φ is one of the chart maps defined above. Show that these are holomorphic embeddings.

(c) Show that $\mathbb{C}P^2$ is compact. *Hint: Show that there is a continuous map from the unit sphere of $\mathbb{C}^3$ to $\mathbb{C}P^2$.*

(2) Let $F(w,z)$ be a polynomial in two variables over $\mathbb{C}$ and consider the set $X \subseteq \mathbb{C}^2$ defined by $F = 0$. Assume that at each point of X, at least one of the partial derivatives $\frac{\partial F}{\partial w}$ or $\frac{\partial F}{\partial z}$ does not vanish. Show that under these assumptions, X is a Riemann surface. In particular, show that if $P(z)$ is a polynomial in one variable with no repeated roots, then $w^2 = P(z)$ defines a Riemann surface.

(3) Let d denote the total degree of F. Show that $\tilde{F}(z_0, z_1, z_2) := {z_0}^d F\left(\frac{z_1}{z_0}, \frac{z_2}{z_0}\right)$ is a homogeneous polynomial of three variables. Show that the equation $\tilde{F} = 0$ defines a compact subset X^* of $\mathbb{C}P^2$. Show that the affine patch ${\varphi_0}^{-1} : \mathbb{C}^2 \to \mathbb{C}P^2$ induces a compactification $X \to X^*$.

(4) Coming back to the specific case $w^2 = P(z)$, how many points at infinty are there: in other words, what is $X^* - X$?

(5) Show that if P has degree 3 and no repeated roots, then X^* is a compact Riemann surface. You may start by showing that, up to linear changes of coordinates, one can consider the equation $w^2 = z(z-1)(z-\lambda)$, where $\lambda \in \mathbb{C} - \{0,1\}$. *NB: In general, X^* can be singular at infinity.*

Exercise 5. Elliptic functions

Let f be a meromorphic funtion on $\mathbb{C}$. A complex number ω is called a *period* of f if $f(z+\omega) = f(z)$ for all $z \in \mathbb{C}$. f is called *doubly periodic* or *elliptic* if it admits two distinct nonzero complex periods ω_1 and ω_2.

(1) Show that the set of periods of f is a $\mathbb{Z}$-module. Show that if f is nonconstant, its set of periods is discrete. By definition, a discrete $\mathbb{Z}$-submodule of $\mathbb{C}$ is a *lattice*. Show that this definition coincides with Exercise 1. (5).

(2) (a) Show that a meromorphic function f is elliptic if and only if there exists a lattice $\Lambda = \omega_1\mathbb{Z} + \omega_2\mathbb{Z} \subseteq \mathbb{C}$, where $\omega_1, \omega_2 \in \mathbb{C}$ are linearly independent over $\mathbb{R}$, such that $f(z+\omega) = f(z)$ for all $z \in \mathbb{C}$ and $\omega \in \Lambda$.

(b) Show that an elliptic function relative to the lattice Λ as above is entirely determined by its restriction to a parallelogram $P_{z_0} := \{z_0 + s\omega_1 + t\omega_2 : 0 \leqslant s,t \leqslant 1\}$. Are there any entire elliptic functions?

(c) Show that an elliptic function is equivalent to a meromorphic function on an elliptic Riemann surface (a complex torus).

(3) (a) Let f be an elliptic function and $P := P_{z_0}$ a parallelogram as in (2)(b). Up to changing z_0, one can assume that the boundary curve ∂P does not contain poles of f (why?). Show that $\int_{\partial P} f = 0$.

(b) Show that an elliptic function has equally many zeros and poles (up to periods), counted with multiplicity. *Hint: you may consider the integral $\int_{\partial P} \frac{f'}{f}$, in other words use the so-called argument principle.* Show that any nonconstant elliptic function is surjective.

(c) Show that the zeros $a_1, \dots, a_n$ and poles $b_1, \dots, b_n$ of an elliptic function satisfy $\sum a_i = \sum b_i$ modulo periods. *Hint: you may consider the integral* $\int_{\partial P} \frac{zf'}{f}$.

Exercise 6. Weierstrass's $\wp$ function

Consider a lattice $\Lambda = \omega_1\mathbb{Z} + \omega_2\mathbb{Z} \subseteq \mathbb{C}$, where $\omega_1, \omega_2 \in \mathbb{C}$ are linearly independent over $\mathbb{R}$.

(1) Let $k > 2$ be a real parameter.

(a) Show that the following series converges:

$$\sum_{(m,n)\in\mathbb{Z}^2-\{(0,0)\}} \frac{1}{\left(m^2+n^2\right)^{\frac{k}{2}}}$$

(b) Let $R > 0$. Show that the following series of functions converges uniformly on $D(0, R)$:

$$\sum_{\omega\in\Lambda, |\omega|\geqslant 2R} \frac{1}{(z-\omega)^{\frac{k}{2}}}$$

(c) Show that the function

$$f_k(z) := \sum_{\omega\in\Lambda} \frac{1}{(z-\omega)^{\frac{k}{2}}}$$

is a well-defined elliptic function. What are its poles?

(2) Weierstrass's $\wp$ function is defined by:

$$\wp(z) := \frac{1}{z^2} + \sum_{\omega\in\Lambda-\{0\}} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}\right)$$

(a) Let $R > 0$. Show that there exists $C > 0$ such that for all $z \in D(0, R)$ and for all $\omega \in \Lambda$ such that $|\omega| > 2R$,

$$\left|\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}\right| \leqslant \frac{C}{\omega^3} .$$

(b) Show that $\wp$ is a meromorphic function on $\mathbb{C}$ and find its poles.

(c) Compute $\wp'$ and show that it is an odd elliptic function.

(d) Show that $\wp$ is an even elliptic function.

(3) Define:

$$g_2 := 60 \sum_{\omega\in\Lambda-\{0\}} \frac{1}{\omega^4} \qquad \text{and} \qquad g_3 := 140 \sum_{\omega\in\Lambda-\{0\}} \frac{1}{\omega^6} .$$

Justify that g_2 and g_3 are well-defined. Study the singularity at 0 of the function $g(z) := \wp'(z)^2 - 4\wp(z)^3 + g_2\wp(z) + g_3$. Conclude that $\wp'(z)^2 = 4\wp(z)^3 + g_2\wp(z) + g_3$ for all $z \in \mathbb{C}$.

(4) (a) Show that $\wp'$ has exactly three zeros: $\frac{\omega_1}{2}$, $\frac{\omega_2}{2}$, and $\frac{\omega_1+\omega_2}{2}$; and that they are all simple.

(b) Show that the map $z \mapsto (\wp(z), \wp'(z))$ defines a biholomorphism between $\mathbb{C}/\Lambda$ and the projective completion of the curve $y^2 = 4x^3 - g_2 x - g_3$ in $\mathbb{C}^2$. *Recall that the projective completion is obtained by adding a single point* (∞, ∞).

(5) Can you reprove the following theorem seen in class?

$$\det \begin{bmatrix} \wp(z_1) & \wp'(z_1) & 1 \\ \wp(z_2) & \wp'(z_2) & 1 \\ \wp(z_3) & \wp'(z_3) & 1 \end{bmatrix} = 0 \quad \text{if and only if} \quad z_1 + z_2 + z_3 = 0 \pmod{\Lambda}\ .$$