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Riemann Surfaces

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Exercises for Chapter 5: Riemann surfaces

Exercise 1. Identity theorem for complex manifolds

State and prove the identity theorem for complex manifolds.

Exercise 2. Whitney embedding theorem for complex manifolds?

(1) Recall the Whitney embedding theorem for compact manifolds.

(2) Recall why a holomorphic function on a compact connected complex manifold is constant.

(3) Recall why there is no Whitney embedding theorem for compact complex manifolds.

(4) A complex manifold that can be embedded in a complex vector space is called a *Stein manifold*. Give examples of Riemann surfaces that are Stein manifolds and others that are not.

Exercise 3. The Riemann sphere

(1) Recall the definition of the Riemann sphere $\hat{\mathbb{C}}$ as a topological space.

(2) Recall how to define a complex structure (i.e. the structure of a Riemann surface) on $\hat{\mathbb{C}}$.

(3) Which entire functions $\mathbb{C} \to \mathbb{C}$ extend to holomorphic functions $\mathbb{C} \to \mathbb{C}$?

(4) Let X be a Riemann surface. Show that a meromorphic function $X \to \mathbb{C}$ is the same as a holomorphic function $X \to \hat{\mathbb{C}}$.

(5) Prove that the group of automorphisms of the Riemann sphere is the group of Möbius transformations, i.e. maps $\hat{\mathbb{C}} \to \hat{\mathbb{C}}$ of the form $z \mapsto \frac{az+b}{cz+d}$, where $a, b, c, d \in \mathbb{C}$ and $ad - bc \neq 0$.

Exercise 4. The complex projective line

(1) Recall the definition of the complex projective line $\mathbb{C}P^1$.

(2) Let $N = [1 : 0]$ and $S = [0 : 1]$ in $\mathbb{C}P^1$. Denote $U_1 = \mathbb{C}P^1 - \{N\}$ and $U_2 = \mathbb{C}P^1 - \{S\}$. Define $z\colon U_1 \to \mathbb{C}$ by $z([z_1 : z_2]) = \frac{z_1}{z_2}$ and define $w\colon U_2 \to \mathbb{C}$ by $w([z_1 : z_2]) = \frac{z_2}{z_1}$. Show that this defines a complex atlas.

(3) Show that $\mathrm{GL}_2(\mathbb{C})$ acts naturally on $\mathbb{C}P^1$. What is the kernel of the action? Derive that $\mathrm{PGL}_2(\mathbb{C}) \hookrightarrow \mathrm{Aut}(\mathbb{C}P^1)$. Write the action of $\mathrm{GL}_2(\mathbb{C})$ on $\mathbb{C}P^1$ in the coordinate z.

(4) Show that $\mathbb{C}P^1$ is biholomorphic to the Riemann sphere via $[z_1 : z_2] \mapsto \frac{z_1}{z_2}$.

(5) Show that as a topological space, $\mathbb{C}P^1 \approx S^3/S^1 \approx S^2$.

(6) Show that $\mathrm{Aut}(\mathbb{C}P^1) \approx \mathrm{PGL}_2(\mathbb{C}) \approx \mathrm{PSL}_2(\mathbb{C})$.

Exercise 5. Complex structures on surfaces of nonnegative Euler characteristic.

List all topological surfaces of finite type that have nonnegative Euler characteristic and find at least one complex structure on each. (*) Can you guess the dimension of the moduli space in each case?

Exercise 6. Elliptic curves

A lattice in $\mathbb{C}$ is an additive subgroup of $\mathbb{C}$ of the form $\Gamma = \mathbb{Z}\tau_1 + \mathbb{Z}\tau_2$, where τ_1 and τ_2 are complex numbers that are linearly independent over $\mathbb{R}$.

(1) Show that of Γ is a lattice, then $\mathbb{C}/\Gamma$ is a Riemann surface. This is called an elliptic curve. What is its genus?

(2) Recall what the automorphisms of $\mathbb{C}$ are.

(3) Show that if f is an automorphism of $\mathbb{C}$ and Γ is a lattice, then $\Gamma' := f(\Gamma)$ is also a lattice, and show that $\mathbb{C}/\Gamma$ and $\mathbb{C}/\Gamma'$ are biholomorphic Riemann surfaces.

(4) Show that any elliptic curve is biholomorphic to $\mathbb{C}/\Gamma$ where Γ is a lattice of the form $\mathbb{Z} + \tau\mathbb{C}$, with $\tau \in \mathbb{H} := \{z \in \mathbb{C} \colon \mathrm{Im}(z) > 0\}$.

(5) Show that elliptic Riemann surfaces are classified up to automorphisms by $\tau \in \mathbb{H}$.

(6) Find the group of automorphisms of an elliptic curve.

(7) Show that elliptic Riemann surfaces are classified by $\mathbb{H}/\mathrm{PSL}_2(\mathbb{Z})$.

Exercise 7. Branched coverings

(1) Let $f \colon X \to Y$ be a holomorphic function between Riemann surfaces. Show that near every point $x \in X$, one can find local complex coordinates on X and Y such that f is written: $f(z) = z^k$ in these coordinates. The integer $k = k_x \in \mathbb{N}$, which depends on $x \in X$, is called the *degree of ramification* of f at x.

(2) Let $f \colon X \to Y$ be a continuous map between Riemann surfaces. Show that the following are equivalent:

(i) f is a proper holomorphic map.

(ii) f is a covering map outside some discrete set $Z \subseteq X$.

When these conditions are satisfied, f is called a *branched* (or *ramified*) covering.

(3) Let $f\colon X \to Y$ be a holomorphic map between Riemann surfaces. Assume Y is compact. Show that f is a branched covering. Show that the number $k_y := \sum_{x \in f^{-1}(y)} k_x$ is independent of $y \in Y$. This number is called the *degree* of f.