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Riemann Surfaces

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Exercises for Chapter 3: Review of manifolds

Exercise 1. The manifold S^1

In this exercise we denote $S^1 = \mathbb{R}/\mathbb{Z}$.

(1) Recall what the topology is on S^1.

(2) Show that there is a unique smooth structure on S^1 such that the canonical projection $\pi\colon \mathbb{R} \to \mathbb{Z}$ is smooth.

(3) Show that $t \mapsto e^{2\pi i t}$ defines a smooth embedding of S^1 in $\mathbb{C} \approx \mathbb{R}^2$.

(4) Find an explicit identification between the tangent bundle $\mathrm{T}S^1$ and the cylinder $S^1 \times \mathbb{R}$.

(5) Let $f(z) = z^2$ for $z \in \mathbb{C}$. Show that f induces a smooth map $S^1 \to S^1$. Describe df.

Exercise 2. Classification of all 1-dimensional manifolds (*)

Classify all 1-dimensional manifolds. *Hint: try to show that any connected* 1*-dimensional manifold is homeomorphic to* $\mathbb{R}$ *or* S^1*. What about if we work in the smooth category?*

Exercise 3. Local coordinates and local frames

Let M be a smooth manifold.

(1) Recall what a system of local coordinates $(x^1, \dots, x^n)$ is.

(2) Given a system of local coordinates $(x^1, \dots, x^n)$, recall the definition of the coordinate 1-forms $\mathrm{d}x^1, \dots, \mathrm{d}x^n$ and the coordinate vector fields $\frac{\partial}{\partial x^i}$.

(3) By definition, a *local frame* consists of locally defined vector fields $X_1, \dots, X_n$ that make up a basis of the tangent space at every point. Is it true that near any point, there exists a local frame of vector fields $X_1, \dots, X_n$ that commute, meaning that the Lie bracket of any two of them is zero? *Hint: take coordinate vector fields.*

(4) (*) Assume that M is equipped with a Riemannian metric. Is it true that near any point, there exists a local frame of vector fields $X_1, \dots, X_n$ that is everywhere orthonormal? Does there exist such a local frame consisting of coordinate vector fields?

Exercise 4. Poincaré duality

Let M be a smooth closed connected oriented manifold of dimension n.

(1) What is the Betti number b_0 equal to?

(2) Explain why the map

$$\int_M : H^n(M,\mathbb{R}) \to \mathbb{R}$$
$$[\alpha] \mapsto \int_M \alpha$$

is well-defined. We will accept without proof that $\int_M$ is a linear isomorphism. *Note: it is a consequence of the Poincaré lemma, it is not hard to prove using partitions of unity, feel free to try it.*

(3) What is the Betti number b_n equal to?

(4) Let $k, l \in \{0, \ldots n\}$. Show that the map

$$B : H^k(M,\mathbb{R}) \times H^l(M,\mathbb{R}) \to H^{k+l}(M,\mathbb{R})$$
$$([\alpha],[\beta]) \mapsto [\alpha \wedge \beta]$$

is well-defined and bilinear. We accept without proof that B is also nondegenerate when $k + l = n$: if $B([\alpha],[\beta]) = 0$ for all $[\beta]$, then $[\alpha] = 0$. *This fact follows from Hodge theory: one can take $\beta = *\alpha$ where $*$ is the Hodge star.*

(5) Derive from the previous questions that for all $k \in \{0, \ldots n\}$, $H^k(M,\mathbb{R})$ is canonically isomorphic to $H^{n-k}(M,\mathbb{R})$. What can you say about the Betti numbers b_k and b_{n-k}?

(6) Show that if M is an odd-dimensional smooth closed connected oriented manifold, then $\chi(M) = 0$.

Exercise 5. Cartan's magic formula (*)

Let M be a smooth manifold. Cartan's formula says that, for any vector field X and for any differential form ω:

$$\mathcal{L}_X(\omega) = i_X(\mathrm{d}\omega) + \mathrm{d}(i_X\omega) .$$

(1) Make sure that you understand the formula. If necessary, lookup *Lie derivative* and *interior product*. Check that both sides of the Cartan formula are the same type of object.

(2) Let us now prove the formula. First show that it is true if ω is a 0-form and if ω is an exact 1-form.

(3) Consider the algebra of differential forms $\Omega(M,\mathbb{R})$ (for the wedge product). Check that for a fixed vector field X, both sides of Cartan's formula are derivations on $\Omega(M,\mathbb{R})$. Conclude.