Manifolds - Lecture 9/13 (part 2)

Chapter 9 : Multilinear algebra

9.1 Tensor product

Let V and W be two finite-dim vector spaces (over $\mathbb{R}$).

Definition
- A pure tensor (or decomposed tensor) is an element of the form
  $v \otimes w$ where $v \in V$, $w \in W$

  "tensor product" is just a notation.

- A tensor is a linear combination of decomposed tensors :
  $\sum \lambda_{ij} \, v_i \otimes w_j$ (finite sum)

- Rules :
  $$(\lambda_1 v_1 + \lambda_2 v_2) \otimes w = \lambda_1 (v_1 \otimes w) + \lambda_2 (v_2 \otimes w)$$
  $$v \otimes (\lambda_1 w_1 + \lambda_2 w_2) = \lambda_1 (v \otimes w_1) + \lambda_2 (v \otimes w_2)$$

We have an obvious vector space structure on the set of tensors by using these rules :

- $v_1 \otimes w_1 + v_2 \otimes w_2$ ✓
- $v \otimes w_1 + v \otimes w_2 = v \otimes (w_1 + w_2)$ ✓
- $\lambda (v \otimes w) = (\lambda v) \otimes w = v \otimes (\lambda w)$ ✓

Definition The vector space of tensors (over V and W) is denoted $V \otimes W$ and called tensor product of vector spaces.

Proposition If $(e_i)_{1 \leq i \leq m}$ is a basis of V
$(e'_j)_{1 \leq j \leq n}$ is a basis of W

Then $(e_i \otimes e'_j)_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}}$ is a basis of $V \otimes W$.

Cor : $\dim (V \otimes W) = (\dim V)(\dim W)$.

More generally, if vector spaces $V_1, \dots, V_N$
we can define the tensor product $V_1 \otimes \dots \otimes V_N$
it has basis $(e_{i_1} \otimes e'_{i_2} \otimes \dots \quad)$

" Concrete definition "

Proposition $V^* \otimes W^* \simeq \mathcal{ML}(V \times W, \mathbb{R})$

space of multilinear maps
$$f : V \times W \longrightarrow \mathbb{R}$$
$$(v, w) \longmapsto f(v, w)$$

Proof $\sum_{i,j} \lambda_{ij} \underset{\substack{\cap \\ V^*}}{\varphi_i} \otimes \underset{\substack{\cap \\ W^*}}{\psi_j} \approx \begin{array}{l} V \times W \longrightarrow \mathbb{R} \\ (v, w) \longmapsto \sum \lambda_{ij} \varphi_i(v) \psi_j(w) \end{array}$

Consequence $\underset{\substack{\wr \\ V^{**}}}{V} \otimes \underset{\substack{\wr \\ W^{**}}}{W} \simeq \mathcal{ML}(V^* \times W^*, \mathbb{R})$

Proposition

- $V^* \otimes W \simeq \mathcal{L}(V, W)$
- $V \otimes \mathbb{R} \simeq V$
- $(V_1 \otimes V_2) \otimes V_3 \simeq V_1 \otimes (V_2 \otimes V_3)$

9.2 Tensor algebra, covariant tensors, contravariant tensors

Definition: $T^k(V) := \underbrace{V \otimes \dots \otimes V}_{k \text{ times}}$

$$T^k(V^*) = \underbrace{V^* \otimes \dots \otimes V^*}_{k \text{ times}} \simeq ML(V \times V \times \dots \times V, \mathbb{R})$$

$$T(V) := \bigoplus_{k=0}^{+\infty} T^k(V) \quad \text{tensor algebra of } V$$

This is an algebra : $\underset{\in T^2(V)}{(v_1 \otimes v_2)} \otimes \underset{\in T^3(V)}{(v_3 \otimes v_4 \otimes v_5)} = \underbrace{v_1 \otimes v_2 \otimes v_3 \otimes v_4 \otimes v_5}_{T^5(V)}$

An element of $T(V)$ is called a <u>purely contravariant tensor</u> over V

$T(V^*)$ is called a <u>purely covariant tensor</u> over V.

9.3 Symmetric and alternating tensors

Now we look at $T(V^*)$ (algebra of multilinear forms on V)

Remark : if $\alpha \in T^k(V^*)$ if $\sigma \in \mathfrak{S}_k$

symmetric group of k elements

then we can define $\sigma \cdot \alpha$ by permuting the entries of α

example $\alpha = \varphi \otimes \psi \in T^2(V^*)$ $\sigma = (1,2) \in \mathfrak{S}_2$

transposition

$\sigma \cdot \alpha = \psi \otimes \varphi$

More generally, if $\alpha \in T^k(V^*) \approx \mathcal{ML}(V \times \dots \times V, \mathbb{R})$

$$\boxed{\sigma \cdot \alpha(v_1, \dots, v_k) = \alpha(v_{\sigma(1)}, \dots, v_{\sigma(k)}).}$$

definition α is symmetric if $\sigma \cdot \alpha = \alpha$ for all $\sigma \in \mathfrak{S}_k$.

is antisymmetric or alternating if $\sigma \cdot \alpha = \mathrm{sign}(\sigma)\,\alpha$ $\forall \sigma \in \mathfrak{S}_k$.

Definition The symmetrization of α is :

$$\mathrm{Sym}(\alpha) := \frac{1}{k!} \sum_{\sigma \in \mathfrak{S}_k} \sigma \cdot \alpha$$

The anti-symmetrization of α is :

$$\mathrm{Alt}(\alpha) := \cancel{\frac{1}{k!}} \sum_{\sigma \in \mathfrak{S}_k} \mathrm{sign}(\sigma)\, \sigma \cdot \alpha$$

in differential geometry

Def The <u>symmetric product</u> of two tensors is

$$\alpha \cdot \beta := \mathrm{Sym}(\alpha \otimes \beta)$$

The <u>wedge product</u> of two tensors is

$$\alpha \wedge \beta := \mathrm{Alt}(\alpha \otimes \beta)$$

Prop $(S(V^*), \cdot)$ is an algebra

space of symmetric tensors

$(\Lambda(V^*), \wedge)$ is an algebra

space of antisymmetric tensors.

Def $\Lambda(V^*)$ is called the <u>exterior algebra</u> of V^*

$$\Lambda(V^*) = \bigoplus_{k=0}^{+\infty} \underbrace{\Lambda^k(V^*)}$$

antisymmetric multilinear forms $V \times V \times \dots V \longrightarrow \mathbb{R}$

also called <u>k- covector</u>.

example : let $V = \mathbb{R}^n$

$\det : V \times V \times \dots \times V \longrightarrow \mathbb{R}$ is an element of $\Lambda^n V^*$

i.e it's a n- covector.

Proposition $\dim \Lambda^k(V^*) = \binom{n}{k} = \dfrac{n!}{k!(n-k)!}$ $n = \dim V$

$= 0$ if $k > n$

Cor : $\Lambda(V^*) = \bigoplus_{k=0}^{n} \Lambda^k V^*$ where $n = \dim V$.

example of wedge product

$\alpha, \beta \in V^* = \Lambda^1 V^*$

$\alpha \wedge \beta \in \Lambda^2 V^* \approx \mathcal{ML}(V \times V, \mathbb{R})$

$$\alpha \wedge \beta (v_1, v_2) = \mathrm{Alt}(\alpha \otimes \beta)(v_1, v_2)$$
$$= (\alpha \otimes \beta - \beta \otimes \alpha)(v_1, v_2)$$
$$= \alpha(v_1)\beta(v_2) - \beta(v_1)\alpha(v_2)$$

$$\boxed{\alpha \wedge \beta (v_1, v_2) = \underbrace{\alpha(v_1)\beta(v_2) - \beta(v_1)\alpha(v_2)}_{2}}$$