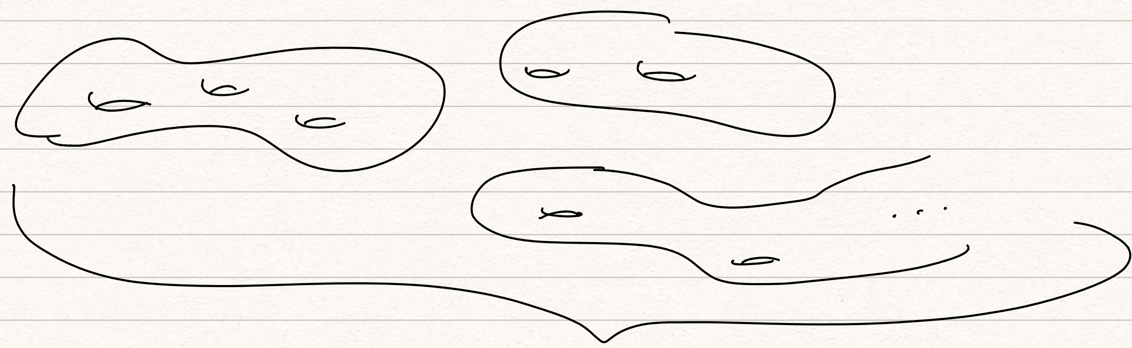
Manifolds - Lecture 2/13

Examples

- $\mathbb{R}^m$ topo manifold of dim m
- $U \subseteq \mathbb{R}^m$ ________________
- More generally, M topological manifold of dimension m then any open subset of M is a ________________.

  In particular, the connected components of any manifold are manifolds of the same dim

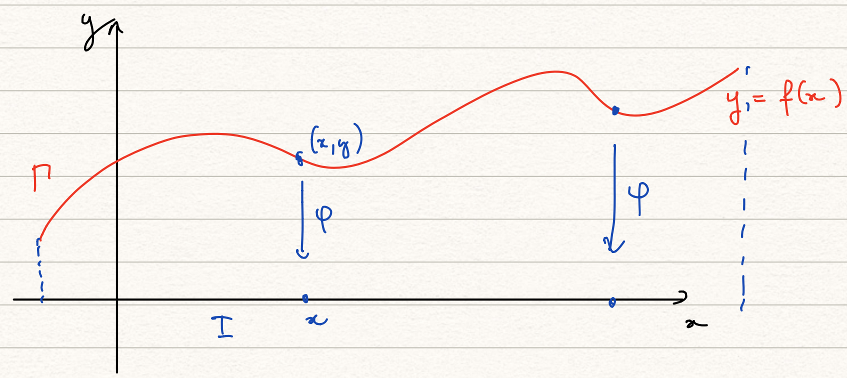


S

- "surface" = connected 2-dimensional topological manifold.

- Let $f : I \subset \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous function

  $\Gamma = \{(x,y) \in \mathbb{R}^2 \mid y = f(x)\} \subseteq \mathbb{R}^2$ graph



Γ is a topological manifold of dimension 1.

Proof: We only use one chart: (U, φ)

$U = \Gamma$, $\varphi : \Gamma \longrightarrow \mathbb{R}$

$(x,y) \longmapsto x$

Need to show: φ is a homeo. Clearly, φ is injective and continuous

Continuous inverse
$$\begin{aligned} I &\longrightarrow \Gamma \\ x &\longmapsto (x, f(x)) \end{aligned}$$
$\blacksquare$

- More generally, $f : V \subseteq \mathbb{R}^m \longrightarrow \mathbb{R}^n$ continuous

Its graph $\Gamma := \{ (x,y) \in \mathbb{R}^m \times \mathbb{R}^n \mid y = f(x) \}$ is a m-dimensional submanifold of $\mathbb{R}^n$.

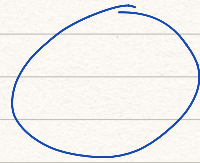
Γ is called a parametrized submanifold and f is called a parametrization.

- Product manifolds

Proposition: If M_1 is a topological manifold m_1 then
M_2 —————— m_2

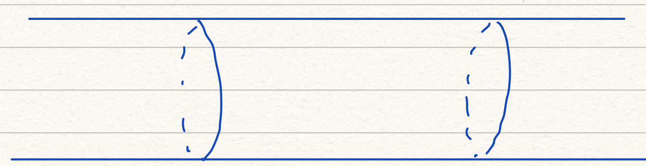
$M_1 \times M_2$ —————— $m_1 + m_2$

example: $M_1 = S^1$



$M_2 = \mathbb{R}$

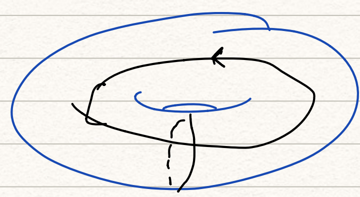
$M_1 \times M_2$



example: $S^1 \times S^1 =: T^2$ is called the torus submanifold of $\mathbb{R}^4$.

$T^2 \approx$

homeo



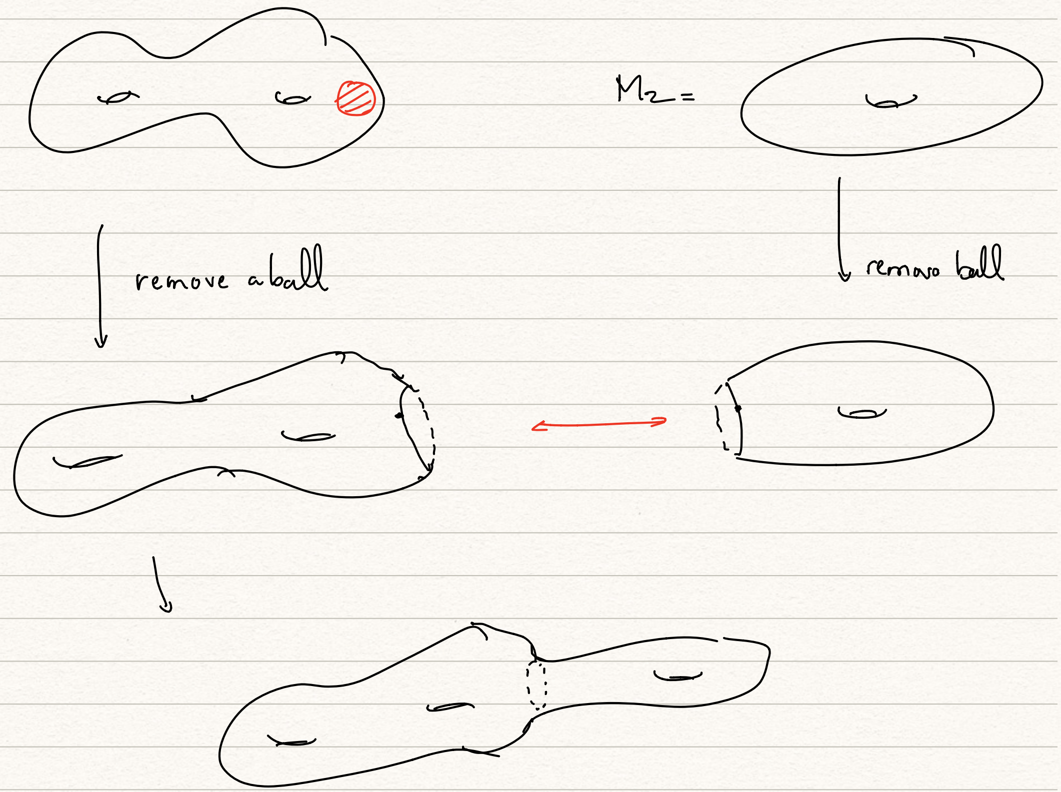
$$T^m := \underbrace{S^1 \times S^1 \times \dots \times S^1}_{m\text{- copies}} \quad m\text{-manifold.}$$

• Connected sums

M_1 and M_2 $\qquad$ $M = M_1 \sqcup M_2 / \sim$ $\qquad$ "gluing"

connected sum :

$M_1 =$ $\qquad$ $M_2 =$



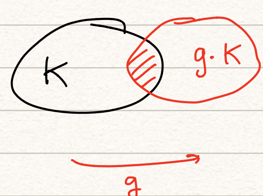
manifolds with boundary

• Quotient manifolds by group actions

<u>Definition</u> : let G be a group acting on a topological space X.

- The action is called <u>faithful</u> : $\forall g \in G \ (\forall x \ g \cdot x = x) \Rightarrow g = e$
- The action is called <u>free</u> : $\forall g \neq e \in G \ \forall x \in X \ \ g \cdot x \neq x$
- The action is called <u>an action by homeos</u> if $\forall g \in G$ $\left(\begin{array}{c} X \to X \\ x \mapsto g \cdot x \end{array} \right)$ is a homeo
- The action is called <u>properly discontinuous</u> :

$\forall K \underset{\text{compact}}{\subseteq} X$, $g \cdot K \cap K = \emptyset$ for all but finitely many $g \in G$.



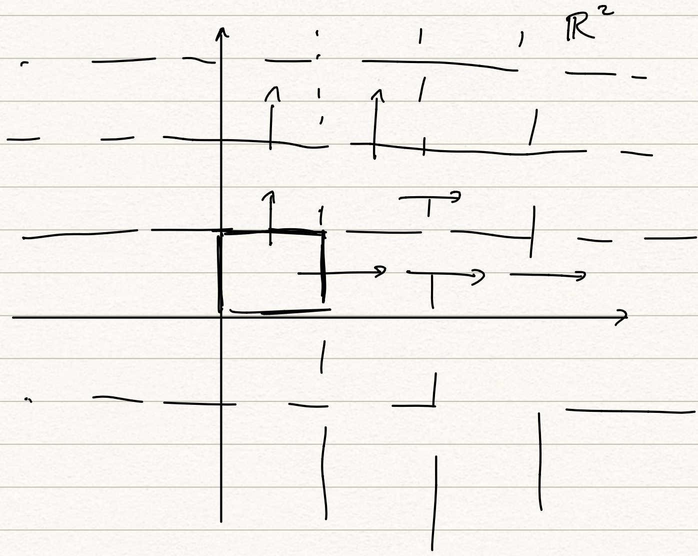
Remark: Here X is assumed Hausdorff and locally compact. (e.g. X is a topo. manifold)

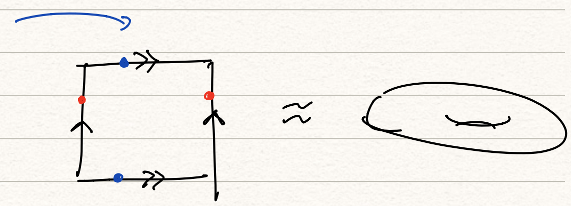
Theorem Let G be a group acting by homeos on a topo manifold M.
If the action is free and properly discontinuous, then M/G is a manifold and $\pi : M \longrightarrow M/G$ is a local homeo.

example: • $\mathbb{R}/\mathbb{Z} \approx S^1$ ($\mathbb{Z}$ acts freely and properly discontinuously on $\mathbb{R}$)

• $\mathbb{Z}^m \subseteq \mathbb{R}^m$ $\quad$ $\mathbb{Z}^m$ acts freely and properly discontinuously on $\mathbb{R}^m$

$\mathbb{R}^m/\mathbb{Z}^m \approx T^m$





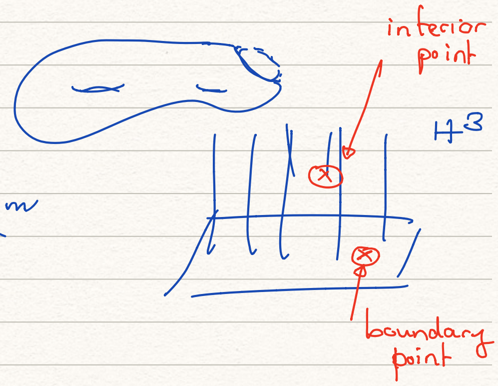
$\mathbb{R}^2/\mathbb{Z}^2 \approx T^2$

Covering maps (algebraic topology).

Proof of theorem above:

1. Show that if $\underset{\text{group}}{G} \curvearrowright \underset{\text{top. space}}{X}$ is a free and wandering action by homeos then $\pi : X \to X/G$ is a local homeo. $\quad \forall x \in X \; \exists U \ni x \; \{g \in G \; gU \cap U \neq \emptyset\}$ is finite
2. Show that if $\underset{\text{group}}{G} \curvearrowright X$ is a properly discontinuous action by homeos, then X/G is Hausdorff.
3. Conclude.

1.4 Topological manifolds with boundary



Let H^m be the closed upper half-space in $\mathbb{R}^m$

$$H^m = \{ x \in \mathbb{R}^m \mid x_m \geq 0 \}.$$

<u>Def</u> A topological manifold with boundary is a topo space X s.t.

(1) X Hausdorff and second-countable

(2) X is locally homeo to H^m.

<u>examples</u>

- H^m
- Any manifold (without boundary) is a "manifold with boundary" (with empty boundary).

$x \in M$ is called a <u>boundary point</u> if there exists a chart $\underset{x}{(U}, \varphi)$ such that $\varphi(x)$ is a boundary point of H^m

$$\varphi : \underset{\substack{\cap \\ M}}{U} \longrightarrow \underset{\substack{\cap \\ H^m}}{V}.$$

The <u>boundary</u> of M is $\partial M := \{ \text{boundary points} \}$

<u>exercise</u>: ∂M is a topological manifold (without boundary).

1.5 Paracompactness and partitions of unity

<u>Proposition</u>: A manifold is always Hausdorff and locally compact.

<u>Definition</u>: A locally compact Hausdorff space X :

- is called <u>σ-compact</u> if it is a countable union of compact sets.
- admits an <u>exhaustion by compact sets</u> if there exists $(K_n)_{n\in\mathbb{N}}$ sequence of compact sets $K_n \subsetneq \operatorname{int}(K_{n+1})$

<u>example</u> $\mathbb{R}^m = \bigcup K_n \qquad K_n = \overline{B(0,n)}$

<u>lemma</u> : Any second-countable, locally compact Hausdorff space admits an exhaustion by compact sets.

<u>Definition</u> A topo space X is called <u>paracompact</u> if any open cover of X admits a locally finite refinement.

- <u>Open cover</u> : $X = \bigcup_{i\in I} U_i$
- <u>Refinement</u> : $X = \bigcup_{j\in J} V_j \qquad \forall j\ \exists i \quad V_j \subseteq U_i$
- <u>locally finite</u> : $\forall x \in X \quad \exists \underset{\text{open}}{U} \ni x$ such that U meets finitely many U_i's.

<u>Theorem</u> : A second-countable, locally compact Hausdorff topo space X is always paracompact.

example : Topological manifolds (with or without boundary).