Manifolds - Lecture 12/13

Chapter 11 - Differential forms

11.3 The exterior derivative

Theorem M smooth manifold.

$\exists!$ linear map $d : \Omega^\bullet(M,\mathbb{R}) \longrightarrow \Omega^\bullet(M,\mathbb{R})$ s.t.

(i) d sends $\Omega^k(M,\mathbb{R})$ into $\Omega^{k+1}(M,\mathbb{R})$.

$$\{0\} \xrightarrow{d} \Omega^0(M,\mathbb{R}) \xrightarrow{d} \Omega^1(M,\mathbb{R}) \xrightarrow{d} \Omega^2(M,\mathbb{R}) \cdots \xrightarrow{d} \Omega^m(M,\mathbb{R}) \xrightarrow{d} \{0\}$$

(ii) On $\Omega^0(M,\mathbb{R}) \approx \mathcal{C}^\infty(M,\mathbb{R})$,
d is the differential of a function (we have seen that if $f \in \mathcal{C}^\infty(M,\mathbb{R})$, $df \in \Omega^1(M,\mathbb{R})$)

(iii) For any $\alpha \in \Omega^k(M,\mathbb{R})$ $\beta \in \Omega^\bullet(M,\mathbb{R})$

$$d(\alpha\wedge\beta) = d\alpha\wedge\beta + (-1)^k \alpha\wedge d\beta$$ ("Leibniz rule")

(iv) $d \circ d = 0$ $\quad (d(d\alpha) = 0)$

Definition d is the exterior derivative

Remark

If $\alpha \in \Omega^0(M,\mathbb{R})$
$\alpha = f \in \mathcal{C}^\infty(M,\mathbb{R})$

$\alpha\wedge\beta = f\beta$

Proof: Uniqueness:

Let $\alpha \in \Omega^k(M,\mathbb{R})$.

Let $(x^1,\dots,x^m)$ be local coordinates.

α can be written $\alpha = \sum \underbrace{\alpha_{i_1, \dots, i_k}}_{\mathcal{C}^\infty(U, \mathbb{R})} dx^{i_1} \wedge \dots \wedge dx^{i_k}$

Claim: $d\alpha$ must be $\underbrace{\sum \underbrace{d(\alpha_{i_1, \dots, i_k})}_{\Omega^1(U, \mathbb{R})} \wedge dx^{i_1} \wedge \dots \wedge dx^{i_k}}$

By the Leibniz rule,

$$d\left(f \, dx^{i_1} \wedge \dots \wedge dx^{i_k}\right) = df \wedge \left(dx^{i_1} \wedge \dots \wedge dx^{i_k}\right) + (-1)^0 f \underbrace{d\left(dx^{i_1} \wedge \dots \wedge dx^{i_k}\right)}_{\text{must be } 0}$$

By (iv) $d(dx^{i_k}) = 0$

By "Leibniz rule" + induction $d\left(dx^{i_1} \wedge \dots \wedge dx^{i_k}\right) = 0$

Existence: expression in red tells me how to define $d\alpha$ in local coordinates.

(+ check consistency when changing coordinates)

Now we need to check (i) (ii) (iii) (iv)

(i) (ii) trivial

(iii) Leibniz rule. straightforward computation.

(iv) $d \circ d = 0$?

$\alpha = f \, dx^{i_1} \wedge \dots \wedge dx^{i_k}$

Want to prove: $d(d\alpha) = 0$?

$d\alpha = df \wedge dx^{i_1} \wedge \dots \wedge dx^{i_k}$

$$d(d\alpha) = d(df) \wedge dx^{i_1} \wedge \dots \wedge dx^{i_k} \qquad (\text{"Leibniz rule"})$$

to conclude, let us show that $d(df) = 0$

In local coordinates $\quad df = \sum_{i=1}^{m} \frac{\partial f}{\partial x^i} \, dx^i$

$$d(df) = \sum_{i=1}^{m} d\left(\frac{\partial f}{\partial x^i}\right) \wedge dx^i$$

$$= \sum_{i=1}^{m} \left(\sum_{j=1}^{m} \frac{\partial^2 f}{\partial x^j \partial x^i} \, dx^j \right) \wedge dx^i$$

$$= \sum_{1 \leq i, j \leq m} \frac{\partial^2 f}{\partial x^j \partial x^i} \, dx^j \wedge dx^i$$

$$= \sum_{1 \leq i < j \leq m} \left(\frac{\partial^2 f}{\partial x^j \partial x^i} \, dx^j \wedge dx^i + \frac{\partial^2 f}{\partial x^i \partial x^j} \, dx^i \wedge dx^j \right)$$

$$+ \sum_{1 \leq i \leq m} \frac{\partial^2 f}{\partial x^{i\,2}} \underbrace{\cancel{dx^i \wedge dx^i}}_{=0}$$

$$= \sum_{1 \leq i < j \leq m} \underbrace{\left(\frac{\partial^2 f}{\partial x^j \partial x^i} - \frac{\partial^2 f}{\partial x^i \partial x^j} \right)}_{0 \text{ by "Schwarz lemma"}} dx^j \wedge dx^i$$

$$= 0 \qquad \blacksquare$$

Example : $M = \mathbb{R}^3$ $\quad (x, y, z)$

$$\Omega^1 \ni \alpha = A \, dx + B \, dy + C \, dz \qquad A \in \mathcal{C}^\infty(\mathbb{R}^3, \mathbb{R})$$

$$d\alpha = dA \wedge dx + dB \wedge dy + dC \wedge dz$$

$$dA = \frac{\partial A}{\partial x} dx + \frac{\partial A}{\partial y} dy + \frac{\partial A}{\partial z} dz$$

$$= \partial_x A \, dx + \partial_y A \, dy + \partial_z A \, dz$$

$$dA \wedge dx = \underline{\partial_y A \, dy \wedge dx} + \partial_z A \, dz \wedge dx$$

$$dB \wedge dy = \ldots$$

$$dC \wedge dz = \ldots$$

$$d\alpha = (\partial_x B - \partial_y A) \, dx \wedge dy + (\partial_y C - \partial_z B) \, dy \wedge dz$$
$$+ (\partial_z A - \partial_x C) \, dz \wedge dx \qquad \in \Omega^2$$

curl ??

$$\beta = P \, dx \wedge dy + Q \, dy \wedge dz + R \, dz \wedge dx$$

$$d\beta = dP \wedge dx \wedge dy + dQ \wedge dy \wedge dz + dR \wedge dz \wedge dx$$

$$= \ldots$$

$$= (\partial_z P + \partial_x Q + \partial_y R) \, dx \wedge dy \wedge dz$$

divergence ??

$\longrightarrow$ Exercise sheet

Further properties:

- Naturality : $f : M \longrightarrow N$

$$f^*(d\alpha) = d(f^*\alpha) \qquad \forall \alpha \in \Omega^k(N, \mathbb{R})$$

- Invariant formula using the Lie bracket (see Lee or Lafontaine)

example for $k=1$

$\alpha \in \Omega^1(M, \mathbb{R})$

$d\alpha \in \Omega^2(M, \mathbb{R})$ is characterized by $\forall X, Y \in \Gamma(TM)$

$$d\alpha(X, Y) = \underline{X} \cdot \underline{\alpha(Y)} + \underline{Y} \cdot \underline{\alpha(X)} - \underline{\alpha([X,Y])}$$

- "Cartan's magic formula"

$$\boxed{\mathcal{L}_X = i_X \circ d + d \circ i_X}$$

$\forall \alpha \in \Omega^k(M, \mathbb{R})$

$\forall X \in \Gamma(TM)$

$$\mathcal{L}_X \alpha = i_X(d\alpha) + d(i_X \alpha)$$

Cor : $\mathcal{L}_X \circ d = d \circ \mathcal{L}_X$

11.4 De Rham cohomology

<u>Definition</u> : $\alpha \in \Omega^k(M,\mathbb{R})$

α is <u>closed</u> if $d\alpha = 0 \iff \alpha \in \operatorname{Ker} d$

α is <u>exact</u> if $\exists \beta \in \Omega^{k-1}(M,\mathbb{R})$ s.t. $\alpha = d\beta \iff \alpha \in \operatorname{Im} d$

<u>Remark</u> $\alpha = d\beta \Rightarrow d\alpha = d(d\beta) = 0$

exact $\Rightarrow$ closed.

<u>Remark</u> $d \circ d = 0 \iff \operatorname{Im} d \subseteq \operatorname{Ker} d$

<u>Definitions</u> : $Z^k(M,\mathbb{R}) = \{\text{closed } k\text{-forms}\} \subseteq \Omega^k$

$B^k(M,\mathbb{R}) = \{\text{exact } k\text{-forms}\} \subseteq \Omega^k$

$B^k \subseteq Z^k$

Consider the quotient $\boxed{\underbrace{H^k_{dR}(M,\mathbb{R})}_{\text{quotient group or quotient space}} = \underbrace{Z^k(M,\mathbb{R}) / B^k(M,\mathbb{R})}_{\text{"de Rham cohomology space"}}}$

<u>Thm</u> : $H^k_{dR}(M,\mathbb{R})$ is finite-dimensional (HARD!)
and only depends on the topology of M.

<u>"Poincaré lemma"</u> : Any closed form is <u>locally</u> exact.

If M is topologically a ball, then $H^k_{dR}(M,\mathbb{R}) = 0$.

De Rham theorem :

$$H^{\bullet}_{dR}(M,\mathbb{R}) \simeq H^{\bullet}(M,\underline{\mathbb{R}}) \simeq H^{\bullet}_{sing}(M,\mathbb{R})$$

sheaf cohomology

Chapter 12 : Integration and Stokes's theorem

Prerequisite : • Multiple integrals in $\mathbb{R}^n$.

• Measure theory and Lebesgue integral : not needed

12.1 Preamble : Integration of differential forms on $\mathbb{R}^m$

Let $U \subseteq \mathbb{R}^m$ be an open set

For $f \in \mathcal{C}^\infty(U,\mathbb{R})$, we can define $\int_U f$

Notation : $\int_U f(x_1,\dots,x_m)\, dx_1 \dots dx_m$

$\int_U f(x)\, d\lambda(x)$ ← "Lebesgue measure"

⚠ f needs to be integrable

For us, we always restrict to easy situation :

- f has compact support in U $\qquad \mathrm{Supp}(f) = \mathrm{cl}(\{x\in U \mid f(x)\neq 0\})$ $\subseteq U$ compact

- $\mathcal{U}$ has compact closure in $\mathbb{R}^m$ and f extends continuously to $\partial\mathcal{U}$.

Let ω be a differential form of top degree on $\mathcal{U}$

$\omega \in \Omega^m(\mathcal{U}, \mathbb{R})$

$\omega = \underbrace{f}_{\in \mathcal{C}^\infty(\mathcal{U},\mathbb{R})} \underbrace{dx^1 \wedge \dots \wedge dx^m}_{\text{basis of } \underbrace{\Lambda^m T^* \mathbb{R}^m}_{1-\text{dim}}}$ where f is a smooth function.

__Remark__

At every point $x \in \mathcal{U}$,

$dx^1 \wedge \dots \wedge dx^m$ is an antisymmetric multilinear map

$\underbrace{\mathbb{R}^m \times \dots \times \mathbb{R}^m}_{m \text{ copies}} \longrightarrow \mathbb{R}$

$dx^1 \wedge \dots \wedge dx^m = \det$

__Definition__ : The integral of ω on $\mathcal{U}$ is

$$\int_{\mathcal{U}} \omega := \int_{\mathcal{U}} f(x) \, d\lambda(x)$$

$$\int_{\mathcal{U}} f(x_1, \dots, x_m) \, dx^1 \wedge \dots \wedge dx^m = \int f(x_1, \dots, x_m) \, dx_1 \dots dx_m$$

__Rem__ : Assume ω is compactly supported

$\operatorname{Supp} \omega \subseteq \mathcal{U}$ is compact.

example : $\omega = x y^2 \, dx \wedge dy$

$$U = (0,1) \times (0,1) \subseteq \mathbb{R}^2$$

$$\omega \in \Omega^2(U, \mathbb{R})$$

$$\int_U \omega := \int_U x y^2 \, dx \, dy$$

$$= \int_{(0,1)\times(0,1)} x y^2 \, dx \, dy$$

$$= \int_0^1 \int_0^1 x y^2 \, dx \, dy$$

$$= \left(\int_0^1 x \, dx \right) \left(\int_0^1 y^2 \, dy \right) = \frac{1}{6} .$$

Proposition : Let $F : U \subseteq \mathbb{R}^m \longrightarrow V \subseteq \mathbb{R}^m$ be an orientation-preserving diffeomorphism

$$\boxed{\int_U F^* \omega = \int_V \omega}$$ for any $\omega \in \Omega^m(V, \mathbb{R})$