Manifolds - Lecture 11/13

Last time : Chapter 10 - Tensor fields on manifolds

10.4 Pullback of covariant tensor fields

$f : M \longrightarrow N$ smooth map

Let A be a covariant tensor field on N

$A \in \Gamma(T^{0,k}(TN))$ $\qquad A_{|p} : T_pN \times \cdots \times T_pN \longrightarrow \mathbb{R}$

Definition The pullback of A by f is the covariant tensor field f^*A defined by $f^*A(v_1, \dots, v_k) := A\big(df(v_1), \dots, df(v_k)\big)$

f^*A is a k-covariant tensor field on M.

Rem : Try to define similarly the pushforward of contravariant tensor fields.

Example : $\omega = xy\, dx \wedge dy + yz\, dx \wedge dz$ on $N = \mathbb{R}^3$

ω is a 2-form on $\mathbb{R}^3$, i.e an alternating 2-covariant tensor field.

$$f : \begin{array}{ccc} M = \mathbb{R}^2 & \longrightarrow & N = \mathbb{R}^3 \\ (x,y) & \longmapsto & (\underbrace{x+y}_{X}, \underbrace{x-y}_{Y}, \underbrace{y}_{Z}) \end{array}$$

$f^*\omega$?

$\omega = XY\, dX \wedge dY + YZ\, dX \wedge dZ$

f is given by $\begin{cases} X = x+y \\ Y = x-y \\ Z = y \end{cases}$

$$dX = d(x+y) = dx + dy$$
$$dY = dx - dy$$
$$dZ = dy$$

$$\begin{aligned} dX \wedge dY &= (dx+dy)\wedge(dx-dy) \\ &= \cancel{dx\wedge dx} - dx\wedge dy + dy\wedge dx - \cancel{dy\wedge dy} \\ &= -2\, dx\wedge dy \end{aligned}$$

$$\begin{aligned} dX \wedge dZ = (dx+dy)\wedge dy &= dx\wedge dy + \cancel{dy\wedge dy} \\ &= dx\wedge dy \end{aligned}$$

$$\begin{aligned} f^*\omega &= (x+y)(x-y)(-2\,dx\wedge dy) + (x-y)\,y\; dx\wedge dy \\ &= \left[-2(x^2-y^2) + (xy-y^2)\right] dx\wedge dy \\ &= \left[-2x^2 + y^2 + xy\right] dx\wedge dy \end{aligned}$$

10.5 Lie derivative of covariant tensor fields

Let M be a smooth manifold

Let X be a vector field on M.

<u>Definition</u> Let A be a k-covariant tensor field on M

$$A \in \Gamma(T^{0,k}(TM)).$$

The <u>Lie derivative of A w.r.t. X</u> is the k-covariant tensor field on M defined by
$$\mathcal{L}_X A = \frac{d}{dt}\Big|_{t=0} \left((\varphi_t^X)^* A \right)$$

Well-defined.

$$\mathcal{L}_X \quad : \quad \Gamma\left(T^{0,k}(TM)\right) \longrightarrow \Gamma\left(T^{0,k}(TM)\right).$$

Properties :

(i) $\mathcal{L}_X$ extends the Lie derivative of functions :

when $k=0$, $\mathcal{L}_X$ coincides with the Lie derivative of functions :

If $f \in \mathcal{C}^\infty(M,\mathbb{R})$ $\quad \mathcal{L}_X f = X(f) = df(X)$.

(ii) $\mathcal{L}_X$ is $\mathbb{R}$-linear. Moreover :

- $\mathcal{L}_{fX}(A) = f\,\mathcal{L}_X(A)$
- $\mathcal{L}_X(fA) = (\mathcal{L}_X f)\,A + f\,\mathcal{L}_X A$ (Leibniz rule)
- $\mathcal{L}_X(A \otimes B) = (\mathcal{L}_X A) \otimes B + A \otimes \mathcal{L}_X B$ (Leibniz rule)
- $\mathcal{L}_X\left(A(X_1,\dots,X_k)\right) = (\mathcal{L}_X A)(X_1,\dots,X_k) + A(\mathcal{L}_X X_1,\dots,X_k) + \dots + A(X_1,\dots,\mathcal{L}_X X_k)$

k-covariant tensor field

vector fields

(Recall $\mathcal{L}_X Y = [X,Y]$ for vector fields)

Chapter 11. Differential forms

11.1 Definition

<u>Definition</u> Let M be a smooth manifold.
A <u>smooth differential form of degree k</u> on M (or a <u>k-form</u> on M) is an alternating k-covariant tensor field.

<u>Rem</u> A k-form is an element of $\Gamma(\Lambda^k T^*M)$

Notation: $\Omega^k(M, \mathbb{R}) = \Gamma(\Lambda^k T^*M)$.

<u>Rem</u> At every $p \in M$ $\Lambda^k T_p^*M$ is a vector space of dimension $\binom{m}{k}$

However $\Omega^k(M, \mathbb{R})$ is typically infinite-dimensional if $k \leq m$
If $k > m$ $\Omega^k(M, \mathbb{R}) = \{0\}$

<u>Recall</u> In local coordinates $(x^1, \dots, x^m)$ on M, any k-form can be written

$$\alpha = \sum_{1 \leq i_1 < i_2 \dots < i_k \leq m} \alpha_{i_1, \dots, i_k} \, dx^{i_1} \dots \wedge dx^{i_k}$$

<u>Examples</u>:
- $k = 0$ 0-forms = smooth functions

  $\Omega^0(M, \mathbb{R}) = \mathcal{C}^\infty(M, \mathbb{R})$

- $k = 1$ $\Omega^1(M, \mathbb{R}) = \Gamma(\Lambda^1 T^*M) = \Gamma(T^*M)$

  1-form = field of covectors

examples: On $\mathbb{R}^3$

$$\alpha = xy^2z^3\, dx - 2z^3\, dy + (x+y)\, dz$$

<u>important example:</u> If $f \in \mathcal{C}^\infty(M, \mathbb{R})$ $\quad df \in \Omega^1(M, \mathbb{R})$

e.g. On $\mathbb{R}^3$ $\quad f(x,y,z) = 2xy^4 + yz^5$

$$df = 2y^4\, dx + 8xy^3\, dy + z^5\, dy + 5yz^4 dz$$

$$df \in \Omega^1(\mathbb{R}^3, \mathbb{R}).$$

- $k=2$ $\quad \alpha = x^2y\, dx \wedge dz - yz\, dy \wedge dz \in \Omega^2(\mathbb{R}^3)$

$$\alpha = e^{x+y}\cos(xy)\, \underbrace{dx \wedge dy}_{\substack{\text{determinant} \\ \text{on } \mathbb{R}^2}} \in \Omega^2(\mathbb{R}^2)$$

- $k=3$

$$\alpha = f(x,y,z)\, dx \wedge dy \wedge dz \in \Omega^3(\mathbb{R}^3)$$

$$\alpha = 0 \in \Omega^3(\mathbb{R}^2)$$

II. Basic operations: Wedge product, Interior product, Pullback

The wedge product defines an operation

$$\begin{aligned} \Omega^k(M,\mathbb{R}) \times \Omega^l(M,\mathbb{R}) &\longrightarrow \Omega^{k+l}(M,\mathbb{R}) \\ (\alpha, \beta) &\longmapsto \alpha \wedge \beta \end{aligned}$$

<u>example</u> $\quad M = \mathbb{R}^3$

$$\alpha = x^2y\, dx \wedge dz - yz\, dy \wedge dz \in \Omega^2(M, \mathbb{R})$$

$$\beta = yz\, dx + z^3\, dy - x\, dz \in \Omega^1(M, \mathbb{R})$$

$$\alpha\wedge\beta = (x^2y\, dx\wedge dz - yz\, dy\wedge dz)\wedge(yz\,dx + z^3\,dy - x\,dz)$$

$$= (x^2y\,dx\wedge dz)\wedge(yz\,dx) + (x^2y\,dx\wedge dz)\wedge(z^3dy)$$
$$+ (x^2y\,dx\wedge dz)\wedge(-x\,dz)$$
$$(-yz\,dy\wedge dz)\wedge(yz\,dx) + (-yz\,dy\wedge dz)\wedge(z^3dy)$$
$$(-yz\,dy\wedge dz)\wedge(-x\,dz)$$

$$(x^2y\,dx\wedge dz)\wedge(yz\,dx) = (x^2y)(yz)\,(dx\wedge dz)\wedge(dx)$$
$$= x^2y^2z\quad dx\wedge dz\wedge dx$$
$$= 0$$

<u>Properties</u>

(i) The wedge product is $\mathcal{C}^\infty(M,\mathbb{R})$ - bilinear :

$$(f\alpha)\wedge\beta = f(\alpha\wedge\beta) = \alpha\wedge(f\beta)$$

(ii) The wedge product is associative

$$(\alpha\wedge\beta)\wedge\gamma = \alpha\wedge(\beta\wedge\gamma)$$

(iii) The wedge product is antisymmetric :

- $\alpha\wedge\alpha = 0$
- More generally, if $\alpha_1, \dots, \alpha_n$ are linearly dependent then $\alpha_1\wedge\dots\wedge\alpha_n = 0$
- $\underbrace{\alpha}_{\text{degree } k}\wedge\underbrace{\beta}_{\text{degree } l} = (-1)^{kl}\,\beta\wedge\alpha$

Proof : Fix a point on M.
Proof boils down to linear algebra.

Definition The algebra of differential forms on M is

$$\Omega^\bullet(M,\mathbb{R}) = \bigoplus_{k=0}^{+\infty} \Omega^k(M,\mathbb{R}) = \bigoplus_{k=0}^{m} \Omega^k(M,\mathbb{R})$$

$(\Omega^\bullet(M,\mathbb{R}), \wedge)$ is a noncommutative algebra
<u>graded algebra</u>

The interior product

Let $\alpha \in \Omega^k(M,\mathbb{R})$ $\quad k>1$

let $X \in \Gamma(TM)$

The interior product of α with X is the $(k-1)$-form denoted $i_X\alpha$ (or $X \lrcorner \alpha$) defined by

$$i_X\alpha(v_1,\dots,v_{k-1}) := \alpha(X, v_1, \dots, v_{k-1})$$

$$i_X\alpha = \alpha(X,\cdot)$$

The interior product defines a map $i_X : \Omega^k(M,\mathbb{R}) \to \Omega^{k-1}(M,\mathbb{R})$

example $\alpha = dx\wedge dy - dy \wedge dz$

$$X = \frac{\partial}{\partial x}$$

$$i_X\alpha(v) = \alpha\left(\frac{\partial}{\partial x}, v\right)$$

$$i_X \alpha(v) = dx \wedge dy\left(\frac{\partial}{\partial x}, v\right) - dy \wedge dz\left(\frac{\partial}{\partial x}, v\right)$$

$$= (dx \otimes dy - dy \otimes dx)\left(\frac{\partial}{\partial x}, v\right) - (dy \otimes dz - dz \otimes dy)\left(\frac{\partial}{\partial x}, v\right)$$

$$= \overbrace{dx\left(\frac{\partial}{\partial x}\right)}^{1} dy(v) - \cancel{dy\left(\frac{\partial}{\partial x}\right)} dx(v)$$
$$- \left(\cancel{dy\left(\frac{\partial}{\partial x}\right)} dz(v) - \cancel{dz\left(\frac{\partial}{\partial x}\right)} dy(v)\right)$$

$$= dy(v)$$

$$\boxed{i_X \alpha = dy}$$

Pullback

$$f : M \longrightarrow N$$

For any $\alpha \in \Omega^k(N, \mathbb{R})$ $\quad f^* \alpha \in \Omega^k(M, \mathbb{R})$.

example : Let $M = N = \{(x,y) \in \mathbb{R}^2 \;\; x > 0\}$

Two systems of local coordinates :
- polar coord (r, θ)
- Cartesian coord (x, y)

Let $\alpha = dx \wedge dy$

How to write α in polar coordinates ?

Transition function $\begin{cases} x = r \cos\theta \\ y = r \sin\theta \end{cases}$ $\qquad F : (r, \theta) \longmapsto (x, y)$

What is $F^* \alpha$?

$$dx = d(r\cos\theta) = dr\cos\theta + r(-\sin\theta)\,d\theta$$
$$dy = d(r\sin\theta) = dr\sin\theta + r\cos\theta\,d\theta$$

$$\begin{aligned}
dx \wedge dy &= (\cos\theta\,dr - r\sin\theta\,d\theta)\wedge(\sin\theta\,dr + r\cos\theta\,d\theta)\\
&= (\cos\theta\,dr)\wedge(r\cos\theta\,d\theta) - (r\sin\theta\,d\theta)\wedge(\sin\theta\,dr)\\
&= r\cos^2\theta\,dr\wedge d\theta - r\sin^2\theta\,d\theta\wedge dr\\
&= r\cos^2\theta\,dr\wedge d\theta + r\sin^2\theta\,dr\wedge d\theta\\
&= r\,dr\wedge d\theta
\end{aligned}$$

M.3 The exterior derivative

So far we have seen:

- derivative of a function: $f \in \overbrace{\mathcal{C}^\infty(M,\mathbb{R})}^{\Omega^0(M,\mathbb{R})} \rightsquigarrow df \in \Omega^1(M,\mathbb{R})$
- Lie derivative $\mathcal{L}_X : \Omega^k(M,\mathbb{R}) \rightsquigarrow \Omega^k(M,\mathbb{R})$, $\alpha \longmapsto \mathcal{L}_X\alpha$

Now: exterior derivative: $d : \Omega^k(M,\mathbb{R}) \longrightarrow \Omega^{k+1}(M,\mathbb{R})$

$k=0$

Fix X

$$\begin{aligned}
\mathcal{C}^\infty(M,\mathbb{R}) &\longrightarrow \mathcal{C}^\infty(M,\mathbb{R})\\
f &\longmapsto X(f)\\
&= \mathcal{L}_X f\\
&= df(X)
\end{aligned}$$