Manifolds. Lecture 10/13

Chapter 9. Multilinear algebra

Last time:
- Tensor product of vector spaces
- Symmetric and alternating tensors

$$T^k(V^*) = \underbrace{V^* \otimes \cdots \otimes V^*}_{k \text{ times}} = \mathcal{ML}(V \times V \times \cdots \times V, \mathbb{R})$$

$S^k(V^*) \subseteq T^k(V^*)$ consisting of symmetric multilinear maps

$\Lambda^k(V^*) \subseteq T^k(V^*)$ ———— alternating ————

<u>example</u>: $\alpha, \beta \in V^*$ <u>Rem</u> $T^1(V^*) = S^1(V^*) = \Lambda^k(V^*) = V^*$

- $\alpha \otimes \beta \in T^2(V^*) = V^* \otimes V^*$ $\alpha \otimes \beta : V \times V \longrightarrow \mathbb{R}$, $(x, y) \longmapsto \alpha(x)\beta(y)$

- $\alpha \cdot \beta \in S^2(V^*)$ $\alpha \cdot \beta = \dfrac{\alpha \otimes \beta + \beta \otimes \alpha}{2}$ $(x, y) \longmapsto \dfrac{\alpha(x)\beta(y) + \alpha(y)\beta(x)}{2}$

  symm product.

- $\alpha \wedge \beta \in \Lambda^2(V^*)$ $\alpha \wedge \beta = \alpha \otimes \beta - \beta \otimes \alpha$ $(x, y) \longmapsto \alpha(x)\beta(y) - \alpha(y)\beta(x)$

  <u>wedge</u> product

  <u>exterior</u>

9.4 Tensors in coordinates

Assume V has a chosen basis $(e_1, \dots, e_m)$.

$$T^{k,l}(V) = \underbrace{(V \otimes V \otimes \dots \otimes V)}_{k \text{ copies}} \otimes \underbrace{(V^* \otimes V^* \dots \otimes V^*)}_{l \text{ copies}}$$

an element of $T^{k,l}(V)$ is called a <u>tensor of mixed type (k,l)</u>

or a <u>k-contravariant l-covariant tensor</u>.

In particular:

$T^{k,0}(V) = T^k(V)$ k-contravariant tensors

$T^{0,l}(V) = T^l(V^*)$ l-covariant tensors.

$S^l(V^*) \subseteq T^{0,l}(V)$ symmetric l-covariant tensors

$\Lambda^l(V^*) \subseteq T^{0,l}(V)$ alternating l-covariant tensors / antisymmetric / skew-symmetric

<u>In coordinates</u>:

- vectors: $v \in V$ $\quad v = \sum_{i=1}^{m} v^i e_i$

- covectors: $\alpha \in V^*$ $\quad \alpha = \sum_{i=1}^{m} \alpha_i e^{i*}$ $\quad \alpha_i = \alpha(e_i)$

- $A \in T^{k,l}(V)$

$$A = \sum_{\substack{1 \leq i_1, \dots, i_k \leq m \\ 1 \leq j_1, \dots, j_l \leq m}} A^{i_1, \dots, i_k}_{ j_1, \dots, j_l} \underbrace{e_{i_1} \otimes \dots \otimes e_{i_k} \otimes e^{j_1 *} \otimes \dots \otimes e^{j_l *}}_{\text{basis of } T^{k,l}(V)}$$

Ricci notation : $A = A^{i_1,\dots,i_k}{}_{j_1,\dots,j_l}$

Elements of $\Lambda^k(V^*)$:

$$\alpha = \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq m} \alpha_{i_1,\dots,i_l} \underbrace{e_{i_1}^* \wedge \dots \wedge e_{i_k}^*}$$

example : $V = \mathbb{R}^3$

$$\alpha : \mathbb{R}^3 \times \mathbb{R}^3 \longrightarrow \mathbb{R}$$

$$((x_1, x_2, x_3), (y_1, y_2, y_3)) \longmapsto x_1 y_3 - x_3 y_1 - 2 x_2 y_3 + 2 x_3 y_2$$

$$\alpha \in \Lambda^2((\mathbb{R}^3)^*)$$

$$\alpha = e_1^* \wedge e_3^* - 2 e_2^* \wedge e_3^*$$

(Recall : $e_3^* \wedge e_1^* = - e_1^* \wedge e_3^*$)

example : Endomorphisms

$$f \in \mathrm{End}(V) = \mathcal{L}(V, V) \simeq V^* \otimes V$$

$f \approx$ mixed tensor of type $(1,1)$

$$f = \sum_{\substack{1 \leq i \leq m \\ 1 \leq j \leq m}} a_i{}^j \, e_i^* \otimes e_j$$

matrix coefficients

example : bilinear maps $f \in \mathcal{ML}(V \times V, \mathbb{R})$

$f \in T^2(V^*)$ 2-covariant-tensor

$$f = \sum_{\substack{1 \leq i \leq m \\ 1 \leq j \leq m}} a_{ij} \, e^{i*} \otimes e^{j*}$$

matrix coefficients

If f is symmetric : $a_{ij} = a_{ji}$

$$f = \sum_{1 \leq i \leq j \leq m} 2 a_{ij} \, e^{i*} \cdot e^{j*}$$

if f is antisymmetric : $a_{ij} = -a_{ji}$

$$f = \sum_{1 \leq i < j \leq m} a_{ij} \, e^{i*} \wedge e^{j*}$$

Chapter 10 - Tensor fields on manifolds

10.1 Tensor product of vector bundles

Recall A smooth vector bundle over a manifold M is a manifold E equipped with a "projection" (smooth surjection) $\pi : E \longrightarrow M$ s.t

$$\begin{tikzcd} \pi^{-1}(U) \subseteq E \arrow[r, "\sim"] \arrow[d, "\pi"] & U \times V \arrow[dl, "p_1"] \\ U \subseteq M & \end{tikzcd}$$

V ← vector space fixed in advance called typical fiber

example : • trivial bundle $E = M \times V \to M$ ← sections = smooth map $M \longrightarrow V$

• tangent bundle $E = TM \to M$ ← sections = vector field

Proposition

• If $E \to M$ is a vector bundle with typical fiber V

then there is a natural "dual vector bundle" $E^* \to M$ with typical fiber V^*

• If $E_1 \to M$ and $E_2 \to M$ are vector bundles with typical fiber V_1 and V_2

then there is a natural "tensor product bundle" $E_1 \otimes E_2 \to M$ with fiber $V_1 \otimes V_2$

Proof : The fiber bundle $\begin{array}{c} E \\ \downarrow \\ M \end{array}$ is completely determined by

the $(U_i)_{i \in I}$ covering of M and the $g_{ij} : U_i \cap U_j \to GL(V)$

To construct E^* : • use the same covering $(U_i)_{i \in I}$

• take the dual maps $g_{ij}^* : U_i \cap U_j \to GL(V^*)$

Recall : $f : V \longrightarrow W$ linear map

$f^* : W^* \longrightarrow V^*$ "dual map" (defined by

"transpose" $f^*(\varphi)(v) = \varphi(f(v))$)

• For $E_1 \otimes E_2$: exercise.

example The dual bundle of TM is called the cotangent bundle

denoted T^*M

$$T^*M = \bigsqcup_{p \in M} T_p^*M$$

dual space to T_pM.

example Endomorphism bundle. Recall : $\mathrm{End}(V) \simeq V^* \otimes V$

$$\mathrm{End}(TM) := T^*M \otimes TM$$

Definition : (Tensor bundles)

$$T^{k,l}(TM) := \underbrace{(TM \otimes TM \otimes \dots \otimes TM)}_{k \text{ copies}} \otimes \underbrace{(T^*M \otimes \dots \otimes T^*M)}_{l \text{ copies}}$$

bundle of k-contravariant l-covariant tensors on M.

At every point $p \in M$

$$T_p^{k,l}(TM) = (T_pM \otimes \cdots \otimes T_pM) \otimes (T_p^*M \otimes \cdots \otimes T_p^*M)$$

$T^k(TM) = T^{k,0}(TM)$ bundle of k-contravariant tensors

$T^l(T^*M) = T^{0,l}(TM)$ bundle of l-covariant tensors

$S^l(T^*M) \subseteq T^l(T^*M)$ bundle of symmetric l-covariant tensors

$\underline{\Lambda^l(T^*M)} \subseteq T^l(T^*M)$ —— alternating ——

all remaining chapters

10.2 Tensor fields

<u>Def</u> : A mixed <u>tensor field</u> of type (k,l) on M is a smooth section of $T^{k,l}(TM)$.

In other words it's a smooth map $A : M \longrightarrow T^{k,l}(TM)$

s.t $A|_p \in T_p^{k,l}(TM)$

notation : $\Gamma(\quad)$

<u>example</u> • A vector field is a $\overbrace{\text{smooth section of}}$ $TM = T^{1,0}(TM)$

i.e A vector field is a 1-contravariant tensor field

(i.e it's a tensor field of type $(1,0)$). $X \in \Gamma(TM)$

• A smooth section of $T^*M = T^{0,1}(TM)$ $\in \Gamma(T^*M)$

is a 1-covariant tensor field also called a <u>1-form</u>.

- A smooth section of $S^k(T^*M)$ $\in \Gamma(S^kT^*M)$
is called a symmetric k-covariant tensor field

- A smooth section of $\Lambda^k(T^*M)$
is called an alternating k-covariant tensor field
or a $\boxed{\text{differential k-form.}}$ $\in \Gamma(\Lambda^k T^*M)$

Famous examples:

- (1,1) tensor field $\approx$ End(TM)
  example: almost complex structure $J \in End(TM)$
  $J^2 = -1$ identity

- Riemannian metric
  $g \in S^2(T^*M)$

- Symplectic form $\omega \in \Lambda^2(T^*M)$ (2-form)

- Riemann curvature tensor: tensor field of type (1,3)

- Type (2,0):
  - Ricci curvature tensor
  - Einstein tensor.

10.3 Tensor fields in coordinates

Assume $(x^1, \dots, x^m)$ are local coordinates

$\rightarrow$ $\left(\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^m}\right)$ basis of TM (at every point)

$\rightarrow$ $(dx^1, \dots, dx^m)$ basis of T^*M (at every point)

Any tensor field of type (k, ℓ) can be written:

$$A = \sum_{\substack{1 \leq i_1, \dots, i_k \leq m \\ 1 \leq j_1, \dots, j_\ell \leq m}} A^{i_1, \dots, i_k}{}_{j_1, \dots, j_\ell} \frac{\partial}{\partial x^1} \otimes \dots \otimes \frac{\partial}{\partial x^k} \otimes dx^1 \otimes \dots \otimes dx^\ell$$

<u>Remark</u>: the coefficients $A^{i_1, \dots, i_k}{}_{j_1, \dots, j_\ell}$ are smooth functions on $U \subseteq M$.

Any differential k-form can be written:

$$\alpha = \sum_{1 \leq i_1 < \dots < i_k \leq m} \alpha_{i_1, \dots, i_k} \, dx^{i_1} \wedge \dots \wedge dx^{i_k}$$

Remark : $\Gamma(T^{k,l}(TM))$ = space of tensor fields of type (k,l)

- is a real vector space
- is a $\mathcal{C}^\infty(M,\mathbb{R})$-module :

if $A \in \Gamma(T^{k,l}(TM))$ and $f \in \mathcal{C}^\infty(M,\mathbb{R})$

$$f A \in \Gamma(T^{k,l}(TM))$$

Proposition : A map $\underbrace{\Gamma(TM) \times \Gamma(TM) \times \dots \times \Gamma(TM)}_{k \text{ copies}} \to \mathcal{C}^\infty(M,\mathbb{R})$

is $\mathcal{C}^\infty(M,\mathbb{R})$-multilinear if and only if it is a tensor field.