Manifolds - Lecture 1/13

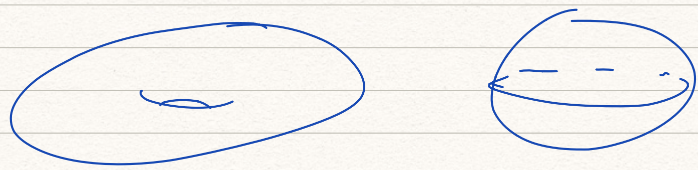
Introduction

What are manifolds?

Generalizations of curves and surfaces

- curves = 1-dim manifolds

- Surfaces = 2-dim manifolds



Terminology

- topological manifolds : special kind of topo. space
- <u>differential manifolds</u> : differentiable manifolds : smooth manifolds
  $\mathcal{C}^\infty$

History

- Euler 1750s
- Gauss 1820s
- Riemann 1850s
- Poincaré 1890s
- Whitney 1930s

Motivation

References

- John Lee's books.
- Lafontaine
- KGB 's lectures notes

Chapter 1. Topological manifolds

1.1 Topological spaces

Lee's Smooth manifolds <u>Appendix A</u>.

<u>Definition</u> Let X be a set. A <u>topology</u> on X is a collection of subsets of X, called <u>open sets</u>, such that:

1) $\emptyset$ and X are open
2) Any union of open sets is open
3) Any finite intersection of open sets is open.

<u>Definition</u>: X a topological is <u>Hausdorff</u> when:

$\forall x, y \in X \quad x \neq y \quad \exists U$ open set $\ni x$, $\exists V$ ——— $\ni y$ $\quad U \cap V = \emptyset$

<u>Definition</u>: X is called <u>second-countable</u> if it has a countable basis of open sets.

<u>Definition</u> $f : X \longrightarrow Y$ is called <u>continuous</u> if the preimage of any open set in Y is an open set of X.

f is a <u>homeomorphism</u> if f is continuous, bijective, and f^{-1} is continuous.

Quotient spaces

X topological space

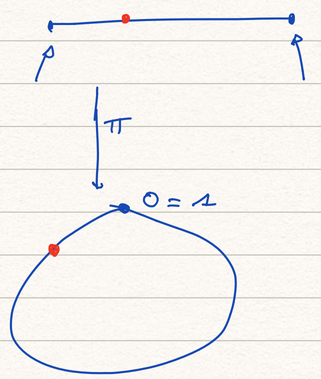
$\sim$ equivalence relation

$X/\sim$ = space of equivalence classes $\qquad \pi : X \longrightarrow X/\sim$

<u>def</u> : $U \subseteq X/\sim$ is <u>open</u> iff $\pi^{-1}(U)$ is open in X

<u>examples</u>

- $X =$



$X = [0,1] \subseteq \mathbb{R}$

$0 \sim 1$

$x \sim x$

- $X =$

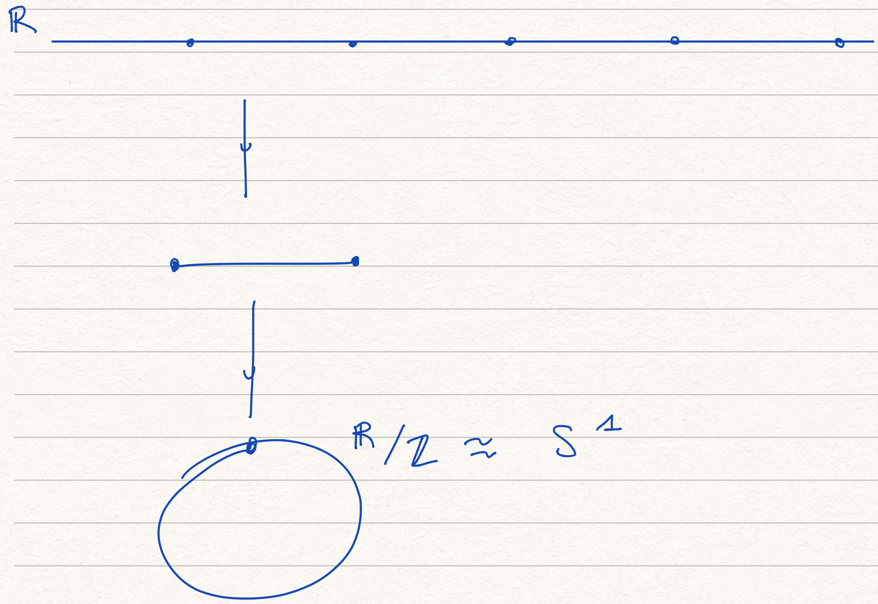
- Group action $G \circlearrowright X$ $\qquad$ partition = space of orbits

$x \sim y \iff \exists g \quad g \cdot x = y$

$X = \mathbb{R}$

$G = \mathbb{Z} \qquad n \cdot x = x + n$

$X/G = \mathbb{R}/\mathbb{Z} = \mathbb{R} \bmod \mathbb{Z}$



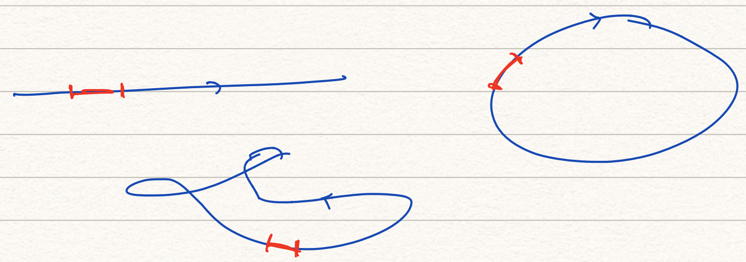
1.2 Topological manifolds

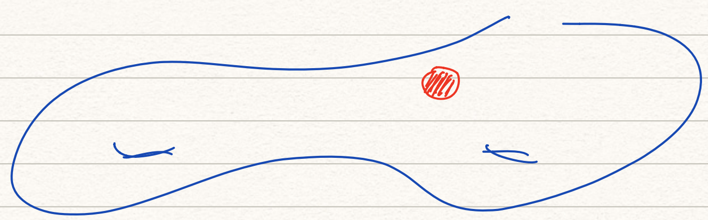
<u>Definition</u> Let $n \in \mathbb{N}$. A <u>topological manifold</u> M is a topological space:

(1) Mild topological restrictions: M is Hausdorff and second-countable

(2) M is locally homeomorphic to $\mathbb{R}^n$:

$\forall x \in M \quad \exists \underset{\text{open}}{U} \ni x$ such that $\exists$ homeomorphism $\varphi : U \longrightarrow \underset{\text{open}}{V} \subseteq \mathbb{R}^n$





Qualities: Manifolds can be compact, connected, etc.

closed manifold: compact manifold (no boundary)

open manifold: noncompact manifold (no boundary)

Exercise Prove that a connect manifold is path-connected.

Dimension:

Theorem (Invariance of domain): $f : U \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^n$ continuous and injective. Then f is open.

Corollary: $U \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^m$ are homeorphic, then $m = n$.

Proof: Assume $m < n$. $f : U \longrightarrow V$ homeomorphism

$$V \subseteq \mathbb{R}^m \subseteq \mathbb{R}^n$$

By the previous theorem f is open.

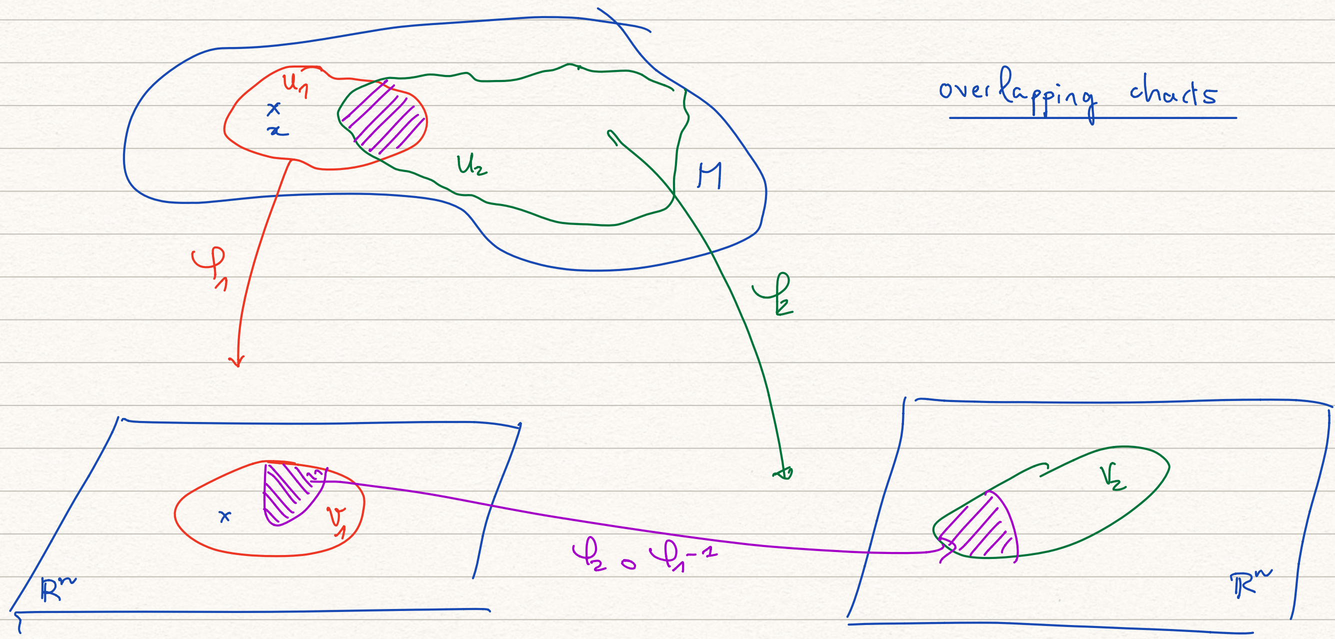
So $f(U) = V$ is open is $\mathbb{R}^n$. However $\mathbb{R}^m \subseteq \mathbb{R}^n$ has empty interior

so $\lightning$ V cannot be open.

Corollary The dimension of any nonempty manifold is uniquely defined.

Atlases

A homeomorphism $\varphi : U \subseteq M \longrightarrow V \subseteq \mathbb{R}^n$ is called a <u>chart</u>.



<u>overlapping charts</u>

The map $\varphi_2 \circ \varphi_1^{-1}$ is well-defined on $\varphi_1(U_1 \cap U_2)$

$$\varphi_2 \circ \varphi_1^{-1} : \underset{\mathbb{R}^n}{\varphi_1(U_1 \cap U_2)} \longrightarrow \underset{\mathbb{R}^n}{\varphi_2(U_1 \cap U_2)}$$

(with $\varphi_1(U_1 \cap U_2) \subset \mathbb{R}^n$ and $\varphi_2(U_1 \cap U_2) \subset \mathbb{R}^n$) is a homeomorphism

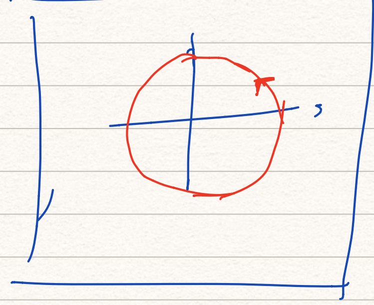
is called a <u>transition function</u>. (<u>change of charts</u>)

A collection of charts $\{ (U_i, \varphi_i)_{i \in I} \}$ that cover M $\left(M = \bigcup_{i \in I} U_i \right)$ is called a <u>topological atlas</u>.

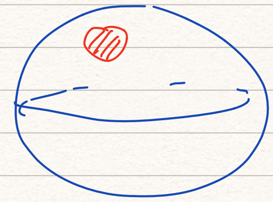
1.3 Examples

- $\mathbb{R}$ is a topological manifold of dimension 1
- Any finite-dimensional vector space is a topo. manifold.

- $S^1 \subseteq \mathbb{R}^2$ is a topo manifold of dim 1.



- $S^n \subseteq \mathbb{R}^{n+1}$ is a top. manifold of dimension n.



- Submanifolds :
  - subset of a manifold which is a manifold
  - More generally, if M is a manifold, an embedded submanifold of M is a manifold N equipped with an embedding $\iota : N \hookrightarrow M$.

    example : $S^n \subseteq \mathbb{R}^{n+1}$