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Chapter 6 Submanifolds

- 6.1 Definition
- 6.2 Characterizations
- 6.3 Tangent bundle to a submanifold
- 6.4 Whitney’s theorems

6.1 Definition

Question: what's a good definition of a submanifold?

Definition

Let N be a smooth n-manifold and let $M \subseteq N$ be a subset. M is a ***smooth submanifold*** of N if $\forall x \in M$, there exists a smooth chart $(U \ni x, \varphi)$ on N s.t. $\varphi(U \cap M) = \varphi(U) \cap \mathbb{R}^m$.

(Roughly speaking, $M \subseteq N$ locally looks like $\mathbb{R}^m \subseteq \mathbb{R}^n$.)

Fact. If M is a smooth submanifold of N, then M is a topo. submanifold of N, and the restriction of the charts (U, φ) as in the definition defines a smooth structure on M.

Proposition (characterization of smooth submanifolds)

Let N be a smooth manifold and let $M \subseteq N$ be a subset. TFAE:

(i) M is a smooth submanifold of N.

(ii) M is a smooth manifold, and the inclusion $\iota \colon M \longrightarrow N$ is a smooth embedding.

Proof.

- (i) $\Rightarrow$ (ii): easy by def. of the smooth structure of M.
- (ii) $\Rightarrow$ (i): Follows from the constant rank theorem.

Extension of the definition.

Definition (embedded and immersed submanifolds)

Let N be a smooth manifold.

- An ***embedded submanifold*** is a smooth manifold M equipped with a smooth embedding $\iota\colon M \hookrightarrow N$.
- An ***immersed submanifold*** is a smooth manifold M equipped with a smooth immersion $\iota\colon M \hookrightarrow N$.

Example. Show that the boundary of the unit square $[0,1] \times [0,1]$ is an embedded submanifold of $\mathbb{R}^2$, but not a smooth submanifold.

Example. Show that the figure "8" in the plane is not an embedded submanifold, but it can be realized as an immersed submanifold.

Examples. See **Chapter 3** for examples of submanifolds, **Chapter 5** for examples of immersed and embedded submanifolds.

Exercise. Show that an embedded submanifold $M \hookrightarrow N$ is properly embedded iff it is a closed subset of N.

6.2 Characterizations

Let us take $N = \mathbb{R}^n$ first.

Theorem

Let $M \subseteq \mathbb{R}^n$ be a subset. The following are equivalent:

(i) M *is a smooth submanifold of dim* m*.*

(ii) M *is locally an embedding of* $\mathbb{R}^m$*:*
$\forall x \in M \, \exists U \ni x \subseteq \mathbb{R}^n$ *and a smooth embed.* $f\colon V \subseteq \mathbb{R}^m \longrightarrow \mathbb{R}^n$ *s.t.* $f(V) = U \cap M$*.*

f *is called a* ***local parametrization*** *of* M*.*

(iii) M *is locally a fiber (level set) of a submersion:*
$\forall x \in M \, \exists U \ni x \subseteq \mathbb{R}^n$ *and* $h\colon U \longrightarrow \mathbb{R}^{n-m}$ *s.t.* $U \cap M = F^{-1}(0)$*.*

(iv) M *is locally the graph of a smooth function:*
$\forall x \in M \, \exists U \ni x \subseteq \mathbb{R}^n$ *and a smooth function* $g\colon V \subseteq \mathbb{R}^m \longrightarrow \mathbb{R}^{n-m}$ *such that* $M \cap U$ *is the graph of* g*, possibly after permuting coordinates.*

Theorem

Let $M \subseteq \mathbb{R}^n$ be a subset. The following are equivalent:

(i) M is a smooth submanifold of dim m.

(ii) M is locally an embedding of $\mathbb{R}^m \to \mathbb{R}^n$.

(iii) M is locally a fiber (level set) of a submersion $\mathbb{R}^n \to \mathbb{R}^{n-m}$.

(iv) M is locally the graph of a smooth function $\mathbb{R}^m \to \mathbb{R}^{n-m}$.

Proof. Essentially, it all follows from the constant rank theorem.

- (i) $\Rightarrow$ (ii): Let φ be a chart as in the def. of a submanifold. Take $(\varphi_{|_M})^{-1}$.
- (i) $\Rightarrow$ (iii): Write $\varphi = (\varphi^1, \ldots, \varphi^n)$. Take $h := (\varphi^{m+1}, \ldots, \varphi^n)$.
- (iii) $\Rightarrow$ (ii), (ii) $\Rightarrow$ (iii), and (ii) $\Rightarrow$ (i): Constant rank theorem.
- (iv) $\Rightarrow$ (iii): Easy (take "$h(x, y) = y - g(x)$")
- (iii) $\Rightarrow$ (iv): Implicit function theorem.

Exercise: Write the details. Good exercise of differential calculus!
(Reference for proof using only the inverse function theorem: [Lafontaine: Chap. 1].)

Remark. Same theorem "in charts" for $N = \mathbb{R}^n$.

Corollary

If $f: M \to N$ is a smooth submersion, then for any $y \in N$, $f^{-1}(y) \subseteq M$ is a smooth submanifold of M of codim. $\dim N$.

Proof. Using charts at $x \in M$ and $y = f(x) \in N$, reduce to open sets of $\mathbb{R}^m$ and $\mathbb{R}^n$ and apply the previous theorem.

Remark. $f^{-1}(y)$ is called a *fiber* of f. It can be empty, it can be disconnected.

Definition

The subset (or a connected component of) $f^{-1}(y) \subseteq M$ is a ***level set*** of f.
$f^{-1}(y)$ is called a ***regular level set*** if f is a submersion at all points of $f^{-1}(y)$.

More generally, we have:

Corollary

Let $f : M \to N$ be a smooth map. Any regular level set of f is a smooth (properly embedded) submanifold.

The previous corollary is often applied when $N = \mathbb{R}$:

Corollary

Let $f : M \to \mathbb{R}$ be a smooth function. Any regular level sets of f is a smooth ***hypersurface*** *[(connected) submanifold of codimension 1].*

Examples of submanifolds defined by submersions.

Conics. Let $P(x, y)$ be a polynomial function of degree 2 in two variables.
The set $C \subseteq \mathbb{R}^2$ defined by $P = 0$ is called a *conic*. If the gradient of P never vanishes (on C), then C is called a regular conic, and is a smooth curve in $\mathbb{R}^2$.

Quadrics. Let $P(x, y, z)$ be a polynomial function of degree 2 in three var.
The set $C \subseteq \mathbb{R}^3$ defined by $P = 0$ is called a *quadric*. If the gradient of P never vanishes (on C), then C is called a regular quadric, and is a smooth surface in $\mathbb{R}^3$.

Go to en.wikipedia.org/wiki/Quadric to see pictures of quadrics.

Projective hypersurfaces. Let $P(x_1, \ldots, x_{m+1})$ be a homogeneous pol.
Assume that the gradient of P does not vanish on $\mathbb{R}^{m+1} - \{0\}$.
Exercise: The equation $P = 0$ defines a smooth submanifold of $\mathbb{R}P^m$.

Examples of submanifolds defined by submersions.

Special linear group. $\mathrm{SL}(n,\mathbb{R})$ is a smooth hypersurface in $\mathrm{M}(n,\mathbb{R})$:
Consider the determinant function $\det\colon \mathrm{M}(n,\mathbb{R}) \to \mathbb{R}$.
It follows that $\mathrm{SL}(n,\mathbb{R})$ is a smooth submanifold of $\mathrm{GL}(n,\mathbb{R})$, so it's a matrix Lie group.

Orthogonal group. $\mathrm{O}(n,\mathbb{R})$ is a smooth submanifold of $\mathrm{M}(n,\mathbb{R})$:
Consider the map $\mathrm{M}(n,\mathbb{R}) \to \mathrm{M}(n,\mathbb{R})$, $M \mapsto {}^tMM$.
It follows that $\mathrm{O}(n,\mathbb{R})$ is a matrix Lie group.

Examples of submanifolds defined by (local) parametrizations .

Smooth immersed curves. Let $\gamma\colon I \to M$ s.t. γ' does not vanish.

Spherical coordinates. (θ, φ) define local parametrizations on S^2.

6.3 Tangent bundle to a submanifold

Proposition

If $M \subseteq N$ is a smooth submanifold, then $\mathrm{T}\,M \subseteq \mathrm{T}\,M$ is a smooth subbundle.

Proof. The differential of the inclusion $\iota\colon M \to N$ is a smooth injective bundle homomorphism $\mathrm{T}\,M \to \mathrm{T}\,N$.

Remark. Similar statement for immersed and embedded submanifolds.

Example (Submanifolds of $\mathbb{R}^n$)

If $M \subseteq \mathbb{R}^n$ is a smooth submanifold, then $\forall x \in M$, $\mathrm{T}_x\,M$ is a vector subspace of $\mathbb{R}^n$.

The affine space through $x \in M$ with underlying vector space $\mathrm{T}_x\,M$ is called the ***affine tangent space*** to $M \subseteq \mathbb{R}^n$.

6.4 Whitney's theorems

First we need the existence of *smooth* bump functions and partitions of unity:

Theorem

Let M be a smooth manifold.

- *$\forall K$ (compact) $\subseteq U$ (open) in M, $\exists$ a smooth bump function f for K supp. in U. (f is a smooth function $M \to \mathbb{R}$ and is a bump function f for K supported in U.)*
- *For any open cover $(U_i)_{i \in I}$, there exists a smooth partition of unity $(\rho_i)_{i \in I}$ subordinate to $(U_i)_{i \in I}$.*

Proof.

- First prove that for any $p \in U$, there exists a compact neighborhood $K_p \subseteq U$ and a smooth bump function f_p for K_p supported in U (use a chart). Then extract a finite subcover from all the K_p and cook up a smooth bump function out of the f_p. *Details: [Lafontaine, Cor. 3.5]*.
- By paracompactness, let $(V_i)_{i \in I}$ be a locally finite refinement of $(U_i)_{i \in I}$. WLOG, we can assume $\overline{V_i}$ is compact and $\subseteq U_i$. Then let f_i be a smooth bump function for $\overline{V_i}$ supported in U_i, and put $\rho_i = \frac{f_i}{\sum_{j \in I} f_j}$. *Details: [Lafontaine, Prop. 6.14]*.

Theorem (Whitney's theorem, easiest version)

Any smooth manifold M admits a smooth embedding into $\mathbb{R}^N$ for some $N \in \mathbb{N}$.

Proof. Same as in Chapter 1, using a smooth partition of unity.

Remark. Does not mean that abstract manifolds are a useless concept!

Theorem (Whitney's theorem, easy version)

In the previous theorem, one can take $N = 2m + 1$ (where $m := \dim M$).
If one only wants an immersion, one can take $N = 2m$.

Proof sketch. By the previous theorem, we can assume M is a submanifold of $\mathbb{R}^N$.
Using Sard's theorem, one can show that $\exists$ a hyperplane $H \subseteq \mathbb{R}^N$ s.t. the orthogonal projection of M to H is an immersion if $N > 2m$ and an embedding if $N > 2m + 1$.
Conclude by induction. *Details: [Lafontaine, Cor 3.8] (or [Lee, Lemma 6.14].)*

Remark. Holds if M is noncompact, holds if M has boundary.

Theorem (Whitney's theorem, strong version)

In the previous theorem, one can take $N = 2m$ (where $m := \dim M$).
If one only wants an immersion, one can take $N = 2m - 1$ (for $m > 1$).

Proof: we admit it. Very hard!

Example. Every surface can be embedded in $\mathbb{R}^4$ and immersed in $\mathbb{R}^3$.

Remark. Even the strong Whitney theorem is not sharp in all dimensions.
For instance, every 3-manifold can be embedded in $\mathbb{R}^5$.
The sharp bound is only known in certain dimensions.

Whitney approximation theorems.

For us, these will be off topic, but let us give a brief mention.
For details, refer to [Lee, end of Chap. 6: "The Whitney approx. theorems"].

Theorem (Whitney approximation theorems)

Let M be a smooth manifold.

- $\forall f : M \to \mathbb{R}^n$ *continuous and* $\forall \varepsilon : M \to \mathbb{R}_{>0}$ *continuous,* $\exists \tilde{f} : M \to \mathbb{R}^n$ *smooth s.t.* $|\tilde{f} - f| < \varepsilon$.
- $\forall f : M \to N$ *continuous where* N *is a smooth manifold,* $\exists \tilde{f} : M \to N$ *smooth s.t.* $\tilde{f}$ *and* f *are homotopic.*

Chapter 7 Vector fields

7.1 Definition

Definition

Let $\pi : E \longrightarrow B$ be a fiber bundle. A ***section of*** E is a map $s : B \longrightarrow E$ s.t. $\pi \circ s = \mathrm{id}_B$.
The space of sections of E is denoted $\Gamma(E)$.

Remarks.

- $s : B \longrightarrow E$ is a section $\Leftrightarrow \forall x \in B \;\; s(x) \in E_x$, where $E_x : \pi^{-1}(x)$ is the fiber over x.
- Henceforth, we only consider smooth fiber bundles and smooth sections.

Definition

Let M be a smooth manifold. A (smooth) ***vector field*** on M is a section of $\mathrm{T}\,M$.
The space of vector fields on M is denoted $\Gamma(\mathrm{T}\,M)$.

Remarks

- A vector field on M is a smooth map $X : M \longrightarrow \mathrm{T}\,M$ s.t. $\forall p \in M, X(p) \in \mathrm{T}_p\,M$.
- We write X_p (or $X_{|_p}$) instead of X_p.

Examples.

Example 1. Let $F\colon U \subseteq \mathbb{R}^m \to \mathbb{R}^m$ be a smooth function.
Define $X\colon U \to \mathrm{T}\,U$ by $X_p = F(p) \in \mathbb{R}^m \approx \mathrm{T}_p\,U$. Then X is a vector field on U.

Remark. In fact, a vector field is always of this form, when looking at it in a chart.

For instance, take $F\colon \mathbb{R}^2 \to \mathbb{R}^2$ defined by:

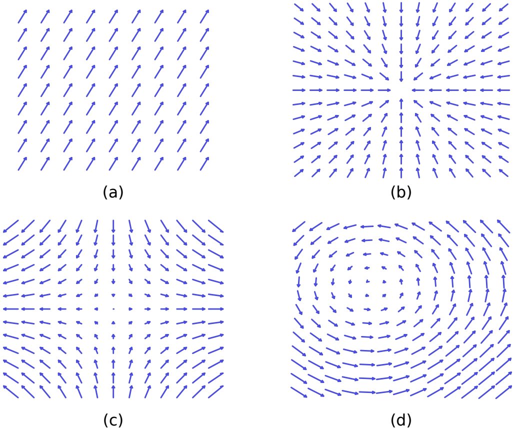
(a) $F(x, y) = (1, 2)$

(b) $F(x, y) = (-x, -y)/\sqrt{x^2 + y^2}$

(c) $F(x, y) = (\cos y, \sin x)$

(d) $F(x, y) = (x, -y)$

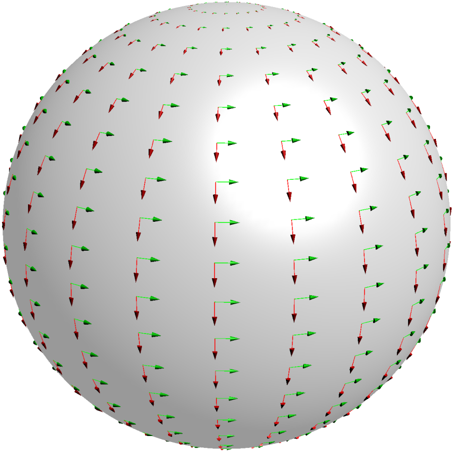
Examples.



Examples.

Example. Two vector fields on the sphere:

- $X_{(x,y,z)} = (-y, x, 0)$
- $Y_{(x,y,z)} = (xz, yz, -x^2 - y^2)$



Examples.

Example: Coordinate vector fields.

Let $(x^1, \dots, x^m)$ be local coordinates on $U \subseteq M$.
Then $\dfrac{\partial}{\partial x^1}, \dots, \dfrac{\partial}{\partial x^m}$ are vector fields on U, called ***coordinate vector fields***.

7.2 Vector fields as derivations

Let X be a vector field on a smooth manifold M. Let $f\colon M \to \mathbb{R}$ be a smooth function.

For every $p \in M$, we can define $\mathrm{d}f_p(X_p) \in \mathbb{R}$.
The function $\mathrm{d}f(X)\colon p \mapsto \mathrm{d}f_p(X_p)$ is a smooth function $M \to \mathbb{R}$.

Recall that a tangent vector $X_p \in \mathrm{T}_p M$ can also be seen as derivation on $C^\infty(M, \mathbb{R})$.
With this point of view, the function $\mathrm{d}f(X)$ is alternatively denoted $\frac{\partial}{\partial X} f$ or $X(f)$ or $X \cdot f$.

Definition

A ***derivation*** on $A := C^\infty(M, \mathbb{R})$ is a $\mathbb{R}$-linear map $D\colon A \to A$ s.t. $\forall f, g \in A$:

$$D(fg) = f\,D(g) + D(f)\,g \qquad \textit{(Leibniz rule)}$$

Proposition

We have a linear isomorphism:

$$\begin{aligned} \Gamma(\mathrm{T}M) &\to \{\textit{Derivations on } C^\infty(M, \mathbb{R})\} \\ X &\mapsto \frac{\partial}{\partial X} \end{aligned}$$

7.3 Pushforward of a vector field

Let X be a vector field on a smooth manifold M.
Let $f: M \to N$ be a smooth function.

For every $p \in M$, we can define $\mathrm{d}f_p(X_p) \in \mathrm{T}_{f(p)} \in \mathrm{T}_{f(p)} N$.
We have a smooth map $\mathrm{d}f(X): M \to \mathrm{T}\, N$.

If f is a diffeo, consider $Y: N \to \mathrm{T}\, N$ defined by $Y = \mathrm{d}f(X) \circ f^{-1}$.

Proposition

*Y is a smooth vector field on N, called **pushforward** of X by f and denoted $f_* X$.*

Proposition

Let $f : M \to N$ be a diffeo. The pushforward map

$$f_* : \Gamma(\mathrm{T}M) \to \Gamma(\mathrm{T}N)$$
$$X \mapsto f_*(X)$$

is a linear isomorphism, whose inverse is $(f^{-1})_*$.

As a map on derivations, the pushforward map is:

$$f_* : \{\textit{Derivations on } C^\infty(M,\mathbb{R})\} \to \{\textit{Derivations on } C^\infty(N,\mathbb{R})\}$$
$$D \mapsto D \circ f^{-1}$$

Proof: Exercise (easy).

7.4 Vector fields in local coordinates