Students evaluation for the “Manifolds” course:
http://evaluation.tu-darmstadt.de/evasys/online.php?pswd=Y5DJH

Chapter 5 Smooth maps

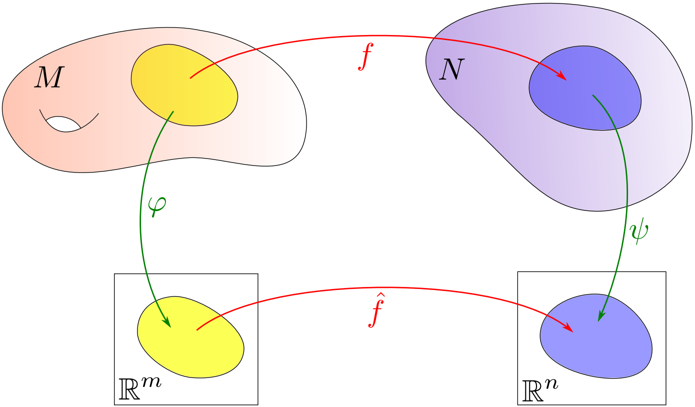
5.1 Recap

What do we know about maps between smooth manifolds so far?

- What it means for a map $f : M \to N$ to be smooth, or a diffeomorphism.
- The differential $\mathrm{d}f$ of a smooth map.
- How to consider f and compute $\mathrm{d}f$ in charts / local coordinates.

Recall that given charts $(U \ni p, \varphi)$ and $(V \ni f(p), \psi)$, we can look at the map $\hat{f} = \psi \circ f \circ \varphi^{-1}$ (***"f in charts"***, or ***"coordinate representation of f"***).

$\hat{f}$ is a smooth map $\mathbb{R}^m \to \mathbb{R}^n$ (well-defined on $\varphi(U)$ provided U is small enough).



M
f
N
φ
ψ
$\hat{f}$
$\mathbb{R}^m$
$\mathbb{R}^n$

Let us review some important facts about smooth maps $\mathbb{R}^m \to \mathbb{R}^n$.

Theorem (Inverse function theorem)

Let $f\colon U \subseteq \mathbb{R}^m \to \mathbb{R}^m$ be a smooth map. Let $x_0 \in U$ s.t. If $\mathrm{d}f_{|x_0}$ is invertible.
Then $\exists V \ni x_0$ s.t. $f_V\colon V \to f(V)$ is a smooth diffeomorphism.

Remark.

- Proof: Banach fixed point theorem for contracting maps between Banach spaces.
- Theorem holds in most other regularity classes C.

Theorem (Implicit function theorem)

Let $f\colon U \subseteq \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}^m$ be a smooth map.
Let $(x_0, y_0) \in U$ s.t. the partial differential $\mathrm{d}_1 f_{|(x_0,y_0)} : \mathbb{R}^m \to \mathbb{R}^m$ is invertible.
Then $\exists V \ni (x_0, y_0)$ and a smooth map $\varphi\colon V \cap \mathbb{R}^m \to \mathbb{R}^n$ s.t., for all $(x, y) \in V$:

$$f(x, y) = 0 \Leftrightarrow y = \varphi(x)$$

Rank of a smooth map and the constant rank theorem:

Let $f\colon U \subseteq \mathbb{R}^m \longrightarrow \mathbb{R}^n$ be a smooth map.

By definition, the ***rank of*** f at $x_0 \in U$ is the rank of the linear map $\mathrm{d}f_{|x_0} : \mathbb{R}^m \longrightarrow \mathbb{R}^n$.

Theorem (Constant rank theorem)

Assume f has constant rank near $x_0 \in U$.

There exists diffeos $\varphi\colon U_1 \ni x_0 \to \varphi(U_1) \subseteq \mathbb{R}^m$ and $\psi\colon V \ni f(x_0) \to \psi(V) \subseteq \mathbb{R}^n$ s.t.

$$\psi \circ f \circ \varphi^{-1} = \mathrm{d}f_{|x_0} .$$

Second version:

$$\psi \circ f \circ \varphi^{-1} = (x_1, \dots, x_m) \mapsto (x_1, \dots, x_r, 0, \dots, 0)$$

where r is the rank of f near p.

Proof. Consequence of Inverse function theorem.
See Lee, Thm 4.12 or Lafontaine, Chap. 1 Exercise 10.

5.2 Rank of a smooth map

Let $f: M \to N$ be a smooth map.

Definition

The ***rank of*** f ***at*** $p \in M$ is the rank of the linear map $\mathrm{d}f_{|_p}: \mathrm{T}_p M \to \mathrm{T}_{f(p)} N$.

Exercise. The rank of f is equal to the rank of the Jacobian matrix of f in any charts.

Proposition. (Upper bound on the rank.) Let $m := \dim M$ and $n := \dim N$.

- By definition, $\mathrm{rk}_p(f) = \dim \mathrm{Im}(\mathrm{d}f_{|_p}) \leqslant \dim \mathrm{T}_{f(p)} N = n$.
- By the rank theorem for linear maps, $\dim \ker(\mathrm{d}f_{|_p}) + \mathrm{rk}_p(f) = m$, so $\mathrm{rk}_p(f) \leqslant m$.

In conclusion, $\mathrm{rk}_p(f) \leqslant \min\{m, n\}$.

Constant rank theorem for maps between smooth manifolds.

Theorem (Constant rank theorem)

Let $f : M \to N$ be a smooth map. Assume f has constant rank $r \in \mathbb{N}$ near $p \in M$. There exists charts (U, φ) at p and (V, ψ) at $f(p)$ s.t. the map $\hat{f} := \psi \circ f \circ \varphi^{-1}$ is:

$$\begin{aligned} \mathbb{R}^m &\to \mathbb{R}^n \\ (x_1, \dots, x_m) &\mapsto (x_1, \dots, x_r, 0, \dots, 0) \end{aligned}$$

Proof. First choose any charts (U_1, φ_1) at p and (V_1, ψ_1) at $f(p)$. Then apply the constant rank theorem (Euclidean version) to $f_1 := \psi_1 \circ f \circ \varphi_1^{-1}$.

Immersions, submersions, local diffeos, critical points.

Let M and N be a smooth manifolds of dim. m and n respectively.
Let $f : M \to N$ be a smooth map and let $p \in M$.

Definition

- If $\mathrm{rk}_p(f) = m$, in other words $\mathrm{d}f_{|_p}$ is injective, f is a ***immersion at*** p.
- If $\mathrm{rk}_p(f) = n$, in other words $\mathrm{d}f_{|_p}$ is surjective, f is a ***submersion at*** p.
- If $\mathrm{rk}_p(f) = m = n$, in other words $\mathrm{d}f_{|_p}$ is bijective, f is a ***local diffeo at*** p.
- If $\mathrm{rk}_p(f) = \min\{m, n\}$, p is a ***regular point*** of f and $f(p)$ is a ***regular value***.
  If $\mathrm{rk}_p(f) < \min\{m, n\}$, p is a ***critical point*** of f and $f(p)$ is a ***critical value***.

Definition

$f : M \to N$ is a ***smooth immersion*** [resp. ***submersion***, resp. ***local diffeo***] if f is smooth and is an immersion [resp. submersion, resp. local diffeo] at p for all $p \in M$.

5.3 Local diffeomorphisms

Let $f\colon M \to N$ be a smooth map.

Proposition (characterization of a local diffeo)

Let $p \in M$. TFAE (the following are equivalent):

(i) $(\mathrm{d}f)_{|p}$ *is invertible.*

(ii) $\exists U \ni x$ *s.t.* $f_{|U}\colon U \to f(U)$ *is a diffeomorphism.*

(iii) There exists a chart (U, φ) *containing* p *and a chart* (V, ψ) *containing* $f(p)$ *such that the map* $\hat{f} := \psi \circ f \circ \varphi^{-1}$ *is the identity.*

Proof. (i) $\Leftrightarrow$ (ii): Follows from inverse function theorem.

- Let $\hat{f} := \psi \circ f \circ \varphi^{-1}$ where (U, φ) is a chart at p and (V, ψ) a chart at $f(p)$.
- $f_{|U}$ is a diffeo $\Leftrightarrow \hat{f} := \psi \circ f \circ \varphi^{-1}$ is a diffeo (because ψ and φ are diffeos)
- $(\mathrm{d}f)_{|p}$ is invertible $\Leftrightarrow (\mathrm{d}\hat{f})_{|\varphi(p)}$ is invertible (by the chain rule)
- Conclude by inverse function theorem.

(ii) $\Leftrightarrow$ (iii): easy.

- If f is a diffeo, take any chart φ at p and put $\psi := \varphi \circ f^{-1}$. (Or: rank theorem.)
- If $\hat{f} := \psi \circ f \circ \varphi^{-1}$ is the identity, then $f = \psi^{-1} \circ \varphi$ is a diffeo.

Definition

A smooth map $f\colon M \to N$ is a ***local diffeo*** if f is a local diffeo at p for all $p \in M$.

Proposition

Let $f\colon M \to N$ be a smooth map. Then f is a diffeomorphism if and only if f is a bijective local diffeomorphism.

Proof: The only thing to show is that if f is a bijective local diffeo, then f^{-1} is smooth. It is enough to show that f^{-1} is locally smooth. Locally, f is a diffeo so f^{-1} is smooth.

Example.

- The map $\mathbb{R} \to S^1$ defined by $t \mapsto e^{it}$ is a local diffeo, but not a global diffeo.
- The induced map $\mathbb{R}/\mathbb{Z} \to S^1$ is a diffeo.

Example.

- Any covering map (fiber bundle with discrete fiber) $\pi\colon M \to B$ is a local diffeo.
- For example, the complex exponential $\exp\colon \mathbb{C} \to \mathbb{C}^*$ is a local diffeo.
- In particular, if G is a group acting freely and properly discontinuously on a manifold M by diffeos, then the projection map $M \to M/G$ is a local diffeo.

5.4 Immersions and embeddings

Let $f: M \longrightarrow N$ be a smooth map.

Definition

- f is an immersion at $p \in M$ if $(\mathrm{d}f)_{|_p}$ is injective.
- f is an immersion if f is an immersion at p for all $p \in M$.

Remark: One must have $m \leqslant n$ (where $m := \dim M$ and $n := \dim N$)

Theorem

f is an immersion at $p \in M$ if and only if there exists a chart (U, φ) at p and a chart (V, ψ) at $f(p)$ such that the map $\hat{f} := \psi \circ f \circ \varphi^{-1}$ is:

$$\begin{aligned} \mathbb{R}^m &\longrightarrow \mathbb{R}^n \\ (x_1, \ldots, x_m) &\longmapsto (x_1, \ldots, x_m, 0, \ldots, 0)\,. \end{aligned}$$

Proof: Particular case of constant rank theorem.

Embeddings.

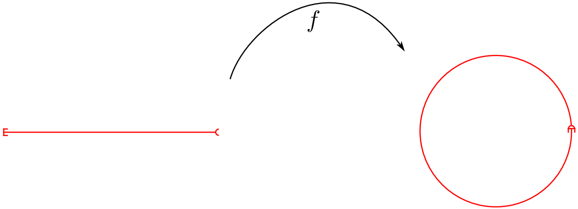
Definition

A map $f : M \to N$ is a ***smooth embedding*** if it is a smooth immersion and a topological manifold.

Recall: ***topological embedding*** = homeomorphism to its image.

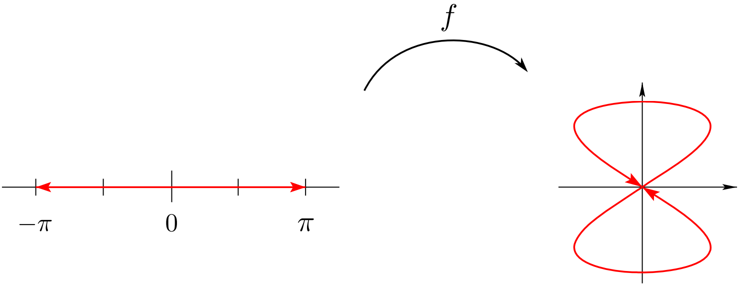
Question: Is a smooth embedding the same thing as an injective smooth embedding?

Answer: YES if M is compact (see Exercise sheet).
NO in general. *Example:* Let $f\colon [0, 2\pi) \to \mathbb{R}^2$, $t \mapsto e^{it}$.



More examples of immersions and embeddings.

Example (Lemniscate). Let $f\colon (-\pi,\pi) \to \mathbb{R}^2$, $t \mapsto (\sin(2t), \sin t)$.



f is an injective immersion, but not an embedding.

More examples of immersions and embeddings.

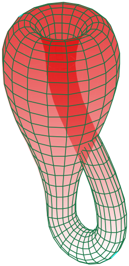
Example (dense curve on torus).

- Let $\alpha \in \mathbb{R} - \pi\mathbb{Q}$ (irrational angle).
  Let $\gamma\colon \mathbb{R} \to \mathbb{C}$, $t \mapsto te^{i\alpha}$. This is an embedded straight line in the plane $\mathbb{C} \approx \mathbb{R}^2$.
- Consider the projection $\pi\colon \mathbb{R}^2 \to T^2 = \mathbb{R}^2/\mathbb{Z}^2$ and the map $\bar{\gamma} = \pi \circ \gamma\colon \mathbb{R} \to T^2$.
  $\bar{\gamma}$ is a smooth curve in T^2, in fact an injective smooth immersion of $\mathbb{R}$ in T^2.
- However, $\bar{\gamma}$ is not embedding. In fact, the image of $\bar{\gamma}$ is dense in T^2.

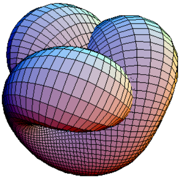
Example (Submanifolds). If $M \subseteq N$ is a submanifold, then the inclusion map $\iota\colon M \to N$ is a smooth embedding. More on submanifolds in Chapter 6.

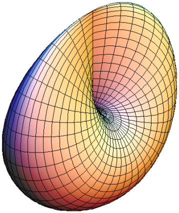
Exercise (immersion = local embedding). Show that a smooth map $f\colon M \subseteq N$ is an immersion iff $\forall x \in M$, $\exists U \ni x$ s.t. $f_{|U}\colon U \to f(U)$ is an embedding.

Example (immersed Klein bottle). A representation of the Klein bottle immersed in $\mathbb{R}^3$:



Example (immersed projective planes).
The cross-cap and Boy's surface are examples of immersions of $\mathbb{R}P^2$ in $\mathbb{R}^3$.





5.5 Submersions

Let $f : M \longrightarrow N$ be a smooth map.

Definition

- f is an submersion at $p \in M$ if $(\mathrm{d}f)_{|_p}$ is surjective.
- f is an submersion if f is an submersion at p for all $p \in M$.

Remark: One must have $m \geqslant n$ (where $m := \dim M$ and $n := \dim N$)

Theorem

f is an immersion at $p \in M$ if and only if there exists a chart (U, φ) at p and a chart (V, ψ) at $f(p)$ such that the map $\hat{f} := \psi \circ f \circ \varphi^{-1}$ is:

$$\begin{aligned} \mathbb{R}^m &\longrightarrow \mathbb{R}^n \\ (x_1, \ldots, x_m) &\longmapsto (x_1, \ldots, x_n, 0, \ldots, 0)\,. \end{aligned}$$

Proof: Particular case of constant rank theorem.

Fundamental example. The projection $pr_1 : M \times N \longrightarrow N$ is a submersion.

More generally:

Theorem

Any smooth fiber bundle $\pi : M \longrightarrow B$ is a submersion.

Proof. Being a submersion is a local propery.
Locally, $\pi \approx$ projection of the first factor $pr_1 : U \times B \longrightarrow B$.

Remark. Conversely, is any submersion a smooth fiber bundle?
Answer: NO. Take any fiber bundle, and remove a point from the total space.
Theorem (Ehresmann.) Any proper submersion is a fiber bundle.

Remark. Given a submersion $\pi : M \longrightarrow B$, what is the relation between smooth maps $N \longrightarrow Y$ and smooth maps $M \longrightarrow Y$ that are constant in the fibers of π?
See [Lee, *Smooth manifolds*].

More examples of submersions.

See Chapter 3 and 4 for examples of fiber bundles:

- Projections $M \times N \to N$
- Cylinder and Möbius strip (over S^1)
- Covering maps: $\mathbb{R} \to S^1$, $\mathbb{R}^2 \to T^2$, $S^n \to \mathbb{R}P^n$, ...
- Hopf fibration $S^3 \to S^2$
- Vector bundles, in part. tangent bundles $\mathrm{T}M \to M$

Remark: If $\pi : M \to B$ is a smooth fiber bundle and $U \subseteq M$ is open, then $\pi_{|U}$ is still a smooth submersion (but not necessarily a fiber bundle).

5.6 Critical points and Sard's theorem

Let M and N be a smooth manifolds of dim. m and n respectively.
Let $f : M \to N$ be a smooth map and let $p \in M$.

Definition

If $\mathrm{rk}_p(f) = \min\{m, n\}$, p is a ***regular point*** of f and $f(p)$ is a ***regular value***.
If $\mathrm{rk}_p(f) < \min\{m, n\}$, p is a ***critical point*** of f and $f(p)$ is a ***critical value***.

Warning.
It is very common to define critical points by $\mathrm{rk}_p(f) < n$ instead of $\mathrm{rk}_p(f) < \min\{m, n\}$.
(That makes the statement of Sard's theorem slightly nicer.)

Remark. A smooth map can have many critical points.
For instance, take a constant map: all points of the domain are critical.
However, it has only one critical value.
More generally, Sard's theorem says that critical values are always rare.

Negligible sets in $\mathbb{R}^m$.

Recall:

- A *rectangle* in $\mathbb{R}^m$ is a product of intervals $R = [a_1, b_1] \times \cdots \times [a_m, b_m]$.
- The *volume* or *measure* of R is $\lambda(R) = |b_1 - a_1| \times \cdots \times |b_m - a_m|$.
- A set $A \subseteq \mathbb{R}^m$ ***has (Lebesgue) measure zero*** or ***is negligible*** if:
  $\forall \varepsilon > 0$, $\exists$ rectangles $(R_n)_{n\in\mathbb{N}}$ s.t. $A \subseteq \bigcup_{n\in\mathbb{N}} R_n$ and $\sum_{n=0}^{+\infty} \lambda(R_n) \leqslant \varepsilon$.

Definition

Let M be a smooth manifold. A subset $A \subseteq M$ is ***negligible*** if, for any smooth chart (U, φ), the set $\varphi(A \cap U)$ is negligible in $\mathbb{R}^m$.

Remark. It is enough to check for the charts of a smooth atlas.
Lemma: If $A \subseteq \mathbb{R}^m$ is negligible and $F\colon U \supset A \to \mathbb{R}^m$ is smooth, then $F(A)$ is negl.

Exercise. The complement of a negligible set is dense.

Theorem (Sard's theorem)

Let $f: M \to N$ be a smooth map.

- *If $m < n$, then $f(M)$ is negligible in N.*
- *In any case, the set of critical values of f is negligible in N.*

Proof. We admit the proof. It is not very difficult but a bit technical.

- The first point is the easiest.
- For the second point when $m = n$, see Lafontaine for a short proof.
- For the secomd point when $m > n$, see Lee or Hirsch.

Corollary

If $M \subseteq N$ is a (embeddded or immersed) submanifold with $m < n$, then M is negligible.

Remark: These results fail in the topological category!

- There exists continuous surjective curves $\gamma\colon [0,1] \to [0,1]^2$ (e.g. Peano).
- There exists embeddings of S^1 in $\mathbb{R}^2$ of positive measure (Osgood curves).

Chapter 6 Submanifolds

6.1 Definition

Question: what's a good definition of a submanifold?

Definition

Let N be a smooth n-manifold and let $M \subseteq N$ be a subset. M is a ***smooth submanifold*** of N if $\forall x \in M$, there exists a smooth chart $(U \ni x, \varphi)$ on N s.t. $\varphi(U \cap M) = \varphi(U) \cap \mathbb{R}^m$.

(Roughly speaking, $M \subseteq N$ locally looks like $\mathbb{R}^m \subseteq \mathbb{R}^n$.)

Fact. If M is a smooth submanifold of N, then M is a topo. submanifold of N, and the restriction of the charts (U, φ) as in the definition defines a smooth structure on M.

Proposition (characterization of smooth submanifolds)

Let N be a smooth manifold and let $M \subseteq N$ be a subset. TFAE:

(i) M is a smooth submanifold of N.

(ii) M is a smooth manifold, and the inclusion $\iota\colon M \to N$ is a smooth embedding.

Proof.

- (i) $\Rightarrow$ (ii): easy by def. of the smooth structure of M.
- (ii) $\Rightarrow$ (i): Follows from the constant rank theorem.

Extension of the definition.

Definition (embedded and immersed submanifolds)

Let N be a smooth manifold.

- An ***embedded submanifold*** is a smooth manifold M equipped with a smooth embedding $\iota\colon M \hookrightarrow N$.
- An ***immersed submanifold*** is a smooth manifold M equipped with a smooth immersion $\iota\colon M \hookrightarrow N$.

Example. Show that the boundary of the unit square $[0, 1] \times [0, 1]$ is an embedded submanifold of $\mathbb{R}^2$, but not a smooth submanifold.

Example. Show that the figure “8” in the plane is not an embedded submanifold, but it can be realized as an immersed submanifold.

Examples. See **Chapter 3** for examples of submanifolds, **Chapter 5** for examples of immersed and embedded submanifolds.

Exercise. Show that an embedded submanifold $M \hookrightarrow N$ is properly embedded iff it is a closed subset of N.

6.2 Characterizations

Let us take $N = \mathbb{R}^n$ first.

Theorem

Let $M \subseteq \mathbb{R}^n$ be a subset. The following are equivalent:

(i) M *is a smooth submanifold of dim* m.

(ii) M *is locally an embedding of* $\mathbb{R}^m$*:*
$\forall x \in M\ \exists U \ni x \subseteq \mathbb{R}^n$ *and a smooth embed.* $f\colon V \subseteq \mathbb{R}^m \longrightarrow \mathbb{R}^n$ *s.t.* $f(V) = U \cap M$.

f *is called a* ***local parametrization*** *of* M.

(iii) M *is locally a fiber (level set) of a submersion:*
$\forall x \in M\ \exists U \ni x \subseteq \mathbb{R}^n$ *and* $h\colon U \longrightarrow \mathbb{R}^{n-m}$ *s.t.* $U \cap M = F^{-1}(0)$.

(iv) M *is locally the graph of a smooth function:*
$\forall x \in M\ \exists U \ni x \subseteq \mathbb{R}^n$ *and a smooth function* $g\colon V \subseteq \mathbb{R}^m \longrightarrow \mathbb{R}^{n-m}$ *such that* $M \cap U$ *is the graph of* g*, possibly after permuting coordinates.*

Theorem

Let $M \subseteq \mathbb{R}^n$ be a subset. The following are equivalent:

(i) *M is a smooth submanifold of dim m.*
(ii) *M is locally an embedding of $\mathbb{R}^m \to \mathbb{R}^n$.*
(iii) *M is locally a fiber (level set) of a submersion $\mathbb{R}^n \to \mathbb{R}^{n-m}$.*
(iv) *M is locally the graph of a smooth function $\mathbb{R}^m \to \mathbb{R}^{n-m}$.*

Proof. Essentially, it all follows from the constant rank theorem.

- (i) $\Rightarrow$ (ii): Let φ be a chart as in the def. of a submanifold. Take $(\varphi_{|_M})^{-1}$.
- (i) $\Rightarrow$ (iii): Write $\varphi = (\varphi^1, \ldots, \varphi^n)$. Take $h := (\varphi^{m+1}, \ldots, \varphi^n)$.
- (iii) $\Rightarrow$ (ii), (ii) $\Rightarrow$ (iii), and (ii) $\Rightarrow$ (i): Constant rank theorem.
- (iv) $\Rightarrow$ (iii): Easy (take "$h(x, y) = y - g(x)$")
- (iii) $\Rightarrow$ (iv): Implicit function theorem.

Exercise: Write the details. Good exercise of differential calculus!
(Reference for proof using only the inverse function theorem: [Lafontaine: Chap. 1].)

Remark. Same theorem “in charts” for $N = \mathbb{R}^n$.

Corollary

If $f : M \to N$ is a smooth submersion, then for any $y \in N$, $f^{-1}(y) \subseteq M$ is a smooth submanifold of M of codim. $\dim N$.

Proof. Using charts at $x \in M$ and $y = f(x) \in N$, reduce to open sets of $\mathbb{R}^m$ and $\mathbb{R}^n$ and apply the previous theorem.

Remark. $f^{-1}(y)$ is called a *fiber* of f. It can be empty, it can be disconnected.
(A connected component of) $f^{-1}(y)$ is called a ***level set*** of f.
$f^{-1}(y)$ is called a ***regular level set*** if f is a submersion at all points of $f^{-1}(y)$.

More generally, we have:

Corollary

Let $f : M \to N$ be a smooth map. Any regular level set of f is a smooth (properly embedded) submanifold.

The previous corollary is often applied when $N = \mathbb{R}$:

Corollary

Let $f : M \to \mathbb{R}$ be a smooth function. Any regular level sets of f is a smooth ***hypersurface*** *[(connected) submanifold of codimension 1].*

Examples of submanifolds defined by submersions.

Conics. Let $P(x, y)$ be a polynomial function of degree 2 in two variables.
The set $C \subseteq \mathbb{R}^2$ defined by $P = 0$ is called a *conic*. If the gradient of P never vanishes (on C), then C is called a regular conic, and is a smooth curve in $\mathbb{R}^2$.

Quadrics. Let $P(x, y, z)$ be a polynomial function of degree 2 in three var.
The set $C \subseteq \mathbb{R}^3$ defined by $P = 0$ is called a *quadric*. If the gradient of P never vanishes (on C), then C is called a regular quadric, and is a smooth surface in $\mathbb{R}^3$.

Go to en.wikipedia.org/wiki/Quadric to see pictures of quadrics.

Projective hypersurfaces. Let $P(x_1, \ldots, x_{m+1})$ be a homogeneous pol.
Assume that the gradient of P does not vanish on $\mathbb{R}^{m+1} - \{0\}$.
Exercise: The equation $P = 0$ defines a smooth submanifold of $\mathbb{R}P^m$.

Examples of submanifolds defined by submersions.

Special linear group. $\mathrm{SL}(n,\mathbb{R})$ is a smooth hypersurface in $\mathrm{M}(n,\mathbb{R})$:
Consider the determinant function $\det\colon \mathrm{M}(n,\mathbb{R}) \to \mathbb{R}$.
It follows that $\mathrm{SL}(n,\mathbb{R})$ is a smooth submanifold of $\mathrm{GL}(n,\mathbb{R})$, so it's a matrix Lie group.

Orthogonal group. $\mathrm{O}(n,\mathbb{R})$ is a smooth submanifold of $\mathrm{M}(n,\mathbb{R})$:
Consider the map $\mathrm{M}(n,\mathbb{R}) \to \mathrm{M}(n,\mathbb{R})$, $M \mapsto {}^tMM$.
It follows that $\mathrm{O}(n,\mathbb{R})$ is a matrix Lie group.

Examples of submanifolds defined by (local) parametrizations .

Smooth immersed curves. Let $\gamma\colon I \to M$ s.t. γ' does not vanish.

Spherical coordinates. (θ,φ) define local parametrizations on S^2.

6.3 Tangent bundle to a submanifold

Proposition

If $M \subseteq N$ is a smooth submanifold, then $\mathrm{T}\,M \subseteq \mathrm{T}\,M$ is a smooth subbundle.

Proof. The differential of the inclusion $\iota \colon M \to N$ is a smooth injective bundle homomorphism $\mathrm{T}\,M \to \mathrm{T}\,N$.

Remark. Similar statement for immersed and embedded submanifolds.

Example (Submanifolds of $\mathbb{R}^n$)

If $M \subseteq \mathbb{R}^n$ is a smooth submanifold, then $\forall x \in M$, $\mathrm{T}_x\,M$ is a vector subspace of $\mathbb{R}^n$.

The affine space through $x \in M$ with underlying vector space $\mathrm{T}_x\,M$ is called the ***affine tangent space*** to $M \subseteq \mathbb{R}^n$.