New resource: Max Planck Institute's online *Manifold Atlas*. Check it out!
http://www.map.mpim-bonn.mpg.de/

Chapter 4 The tangent bundle

How to define tangent vectors to an abstract manifold?

- Idea 1: A tangent vector is the velocity of a curve. $u = \gamma'(0)$
- Idea 2: A tangent vector is a direction to take the derivative of functions. $f \in C^\infty(M, \mathbb{R}) \mapsto \frac{\partial f}{\partial u} = \mathrm{d}f(u)$.

4.1 Tangent vectors as velocities

Let M be a smooth manifold. A ***(smooth) curve*** on M is a smooth map $\gamma\colon I \to M$, where $I \subseteq \mathbb{R}$ is an interval.

Let $p \in M$. Let us say that two smooth curves $\gamma_i\colon I_i \to M$ ($i \in \{1, 2\}$) s.t. $\gamma_i(0) = p$ *have same velocity at* p if $(\varphi \circ \gamma_1)'(0) = (\varphi \circ \gamma_2)'(0)$ for some/any chart (U, φ).

Definition

A ***tangent vector*** to M at p is an equivalence class of smooth curves “having same velocity at p”.

The ***tangent space*** to M at p is the set of tangent vectors at p, denoted $\mathrm{T}_p M$.

Let M be a smooth manifold of dim. m, and let $p \in M$. Consider a chart (U, φ) containing p.

For any $v = [\gamma] \in \mathrm{T}_p M$, the vector $(\varphi \circ \gamma)'(0) \in \mathbb{R}^m$ is well-defined.
Let us denote it $\varphi_* v$ and call it *image v in the chart (U, φ)*.

Proposition

$\mathrm{T}_p M$ is a vector space of dim. m, and for any chart (U, φ) the map $v \mapsto \varphi_* v$ is a linear isomorphism $\mathrm{T}_p M \xrightarrow{\sim} \mathbb{R}^m$.

Proof.

- It is clear that φ_* is bijective.
- Want to show: the linear structure defined by $\varphi_* \colon \mathrm{T}_p M \xrightarrow{\sim} \mathbb{R}^m$ is ind. of φ.
- By chain rule, $\psi_* \circ (\varphi_*)^{-1} = \mathrm{d}F_{|0} : \mathbb{R}^m \to \mathbb{R}^m$ where $F = \psi \circ \varphi^{-1}$.
- Since F is a diffeomorphism (transition function), $\mathrm{d}F_{|0}$ is a linear isomorphism.

□

Remark. If $M = U \subseteq \mathbb{R}^m$, then $\mathrm{T}_p M \approx \mathbb{R}^m$.

4.2 The tangent bundle

Let $\mathrm{T}\,M := \bigsqcup_{p \in M} \mathrm{T}_p\,M$.

There is a "canonical projection" $\pi\colon\ \mathrm{T}\,M \to M$ such that $v \in \mathrm{T}_p\,M \mapsto p$.

In other words, $\pi^{-1}(\{p\}) = \mathrm{T}_p\,M$.

Proposition

*$\pi\colon\ \mathrm{T}\,M \to M$ is a smooth fiber bundle, called the **tangent bundle** to M.*

Proof.

We show simultaneously that $\mathrm{T}\,M$ is a smooth manifold, and that $\pi\colon\ \mathrm{T}\,M \to M$ is a smooth fiber bundle.

First let us discuss some generalities about fiber bundles.

Generalities on fiber bundles.

A top. space X is a ***fiber bundle*** over B with typical fiber F if it is equipped with a projection $\pi\colon X \longrightarrow B$ s.t. $\forall x \in B\ \exists U \ni x$

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\exists\varphi \text{ homeo}} & U \times F \\ & \searrow^{\pi} \quad \swarrow_{\mathrm{pr}_1} & \\ & U & \end{array}$$

Equivalently, there exists (U_i, φ_i) where $(U_i)_{i\in I}$ is a covering of B and $\varphi_i\colon \pi^{-1}(U_i) \to U_i \times F$ is a homeomorphism s.t. diagram commutes.

When U_i and U_j intersect (let $U_{ij} := U_i \cap U_j$), we have the following situation:

$$\begin{array}{ccc} U_{ij} \times F & \xrightarrow{\varphi_j \circ \varphi_i^{-1}} & U_{ij} \times F \\ & \searrow^{\mathrm{pr}_1} \quad \swarrow_{\mathrm{pr}_1} & \\ & U & \end{array}$$

The diagram commutes $\Leftrightarrow \varphi_j \circ \varphi_i^{-1}(x, y) = (x, g_{ij}(x, y))$ for some map $g_{ij}\colon U_{ij} \times F \to F$. Think of g_{ij} as a map $U_{ij} \to \mathrm{Homeo}(F)$ $(x \mapsto g_{ij}(x, \cdot))$

Remark. The data of B (base), $(U_i)_{i\in I}$ (open cover of B), F (typical fiber), and $(g_{ij})_{i,j\in I}$ s.t. $g_{ij} \circ g_{jk} = g_{ik}$ is enough to recover the fiber bundle.

Definition

Let $G \leqslant \mathrm{Homeo}(F)$ be a subgroup.
If $g_{ij}(x) \in G$ for all $x \in U_{ij}$ and for all $i, j \in I$, the fiber bundle is called a ***G-bundle***.

Notable examples:

- $G = \{\mathrm{id}\}$. $M \approx B \times F$: ***trivial bundle***.
- F is discrete: ***covering map***.
- $F = G$ is a group acting on itself by multiplication: ***principal bundle***.
- F is a vector space and $G = \mathrm{GL}(F)$: ***vector bundle***.
- B, F are smooth manifolds and $g_{ij} : U_{ij} \times F \to F$ are smooth: ***smooth fiber bundle***. In part., $G = \mathrm{Diffeo}(F)$.

Proposition

In the last case (smooth fiber bundle), there exists a unique smooth structure on M such that each map $\varphi_i : \pi^{-1}(U_i) \to U_i \times F$ is a diffeo.

Back to the tangent bundle.

Proposition

$\pi\colon \mathrm{T}M \to M$ is a smooth vector bundle.

Proof.

Let (U_i, φ_i) be a smooth atlas on M.

Consider the map $\Phi_i\colon \pi^{-1}(U_i) \subseteq \mathrm{T}M \to U_i \times \mathbb{R}^m$ defined by $\Phi_i(v) = (\pi(v), (\varphi_i)_* v)$.

Each Φ_i is bijective, and when U_i and U_j intersect we have:

$$\begin{aligned}\Phi_j \circ (\Phi_i)^{-1}\colon (p, v) \mapsto & \left(p, (\varphi_j)_* \left[(\varphi_i)_*\right]^{-1} v\right) \\ & = \left(p, \mathrm{d}F_{ij}|_{\varphi_i(x)} v\right)\end{aligned}$$

where $F_{ij} = \varphi_j \circ \varphi_i^{-1}$ is the transition function.

The map $g_{ij}\colon (x, v) \mapsto \mathrm{d}F_{ij}|_{\varphi_i(x)} v$ is smooth and linear in v, we conclude that $\mathrm{T}M$ is a smooth vector bundle. □

4.3 Differential of a function

Let $f\colon M \longrightarrow N$ be a smooth function.

We would like to define the *differential* (or *derivative*) $\mathrm{d}f\colon \mathrm{T}M \longrightarrow \mathrm{T}N$
s.t. for all $p \in M$, the restriction $\mathrm{d}f_{|\mathrm{T}_p M}$ is a linear map $\mathrm{T}_p M \longrightarrow \mathrm{T}_{f(p)} N$.

Observe that for any smooth curve $c\colon I \longrightarrow M$ through $p \in M$, we have a smooth curve $f \circ c\colon I \longrightarrow N$ through $f(p) \in N$.

Definition

The ***differential*** (or ***derivative***) of f is the map $\mathrm{d}f\colon \mathrm{T}M \longrightarrow \mathrm{T}N$ induced by the map $c \mapsto f \circ c$ on smooth curves.

Remark. For all $p \in M$, $\mathrm{d}f_{|\mathrm{T}_p M}$ is a linear map $\mathrm{T}_p M \longrightarrow \mathrm{T}_{f(p)} N$:
$\mathrm{d}f$ is a ***homomorphism of vector bundles.***

$$\begin{array}{ccc} \mathrm{T}M & \xrightarrow{\ \mathrm{d}f\ } & \mathrm{T}N \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ M & \xrightarrow{\ f\ } & N \end{array}$$

Terminology. The differential of f is also called:
derivative of f, *linear tangent map* to f, or *pushforward*.

Notations:

$$\mathrm{d}f|_p(v) \qquad \mathrm{d}f_p v \qquad \mathrm{d}f(p)v$$

$$\mathrm{D}f|_p(v) \qquad \mathrm{D}f_p(v) \qquad \mathrm{D}f(p)v \qquad \mathrm{D}_p f(v) \qquad \mathrm{D}_v f$$

$$f_* v \qquad \frac{\partial f}{\partial v} \qquad v \cdot f$$

Examples:

- Let $M = U \subseteq \mathbb{R}^m$ and $N = \mathbb{R}^n$.
  Then $\mathrm{d}f|_p : \mathrm{T}_p M \approx \mathbb{R}^m \longrightarrow \mathrm{T}_{f(p)} N \approx \mathbb{R}^n$ is the usual differential.
- Let $\varphi : U \subseteq M \longrightarrow \mathbb{R}^m$ be a smooth chart. Then $\mathrm{d}\varphi = \varphi_*$.
- Let $f : M \longrightarrow N$ be constant. Then $\mathrm{d}f \equiv 0$.
- Let $f : M \longrightarrow N = \mathbb{R}$. Then $\mathrm{d}f|_p : \mathrm{T}_p M \longrightarrow \mathrm{T}_{f(p)} \mathbb{R} \approx \mathbb{R}$.
  In other words, $\mathrm{d}f|_p \in \mathrm{T}_p^* M$.

4.4 Tangent vectors as derivations

Idea 2: A tangent vector is a direction to take the derivative of functions.

Let M be a smooth manifold and let $p \in M$.

Consider the algebra $A = C^\infty(M, \mathbb{R})$.
(Better: ***localize*** *by taking $A = C^\infty_p(M, \mathbb{R}) := \{$germs of smooth functions at $p\}$.)*

For any $v \in \mathrm{T}_p M$, consider the map $\frac{\partial}{\partial v} : A \to \mathbb{R}$ defined by $f \mapsto \mathrm{d}f_p(v)$.

Definition

A ***derivation at*** p is a linear map $D \colon A \to \mathbb{R}$ that satisfies the *Leibniz rule*:

$$D(fg) = D(f)g(p) + f(p)D(g)$$

Theorem

We have a linear isomorphism:

$$\mathrm{T}_p M \to \{\textit{Derivations at } p\}$$
$$v \mapsto \frac{\partial}{\partial v}$$

Proof sketch.

- By using a bump function, we can work on an open set U containing p.
- By using a chart, we can assume $U \subseteq \mathbb{R}^m$.
- Injectivity and linearity is easy.
- Surjectivity: use Taylor expansion.

□

Remark. If one defines tangent vectors as derivations, the definition of the differential of a function is trivial: $\mathrm{d}f_p(v) := v(f)$.

Henceforth, we identify tangent vectors and derivations. $v \leftrightarrow \frac{\partial}{\partial v}$

4.5 Tangent vectors and differentials in local coordinates

Let $(x^1, \dots, x^m)$ be local coordinates on M.

Recall: This means that there is a chart (U, φ) such that $\varphi = (x^1, \dots, x^m)$.
For each $i \in \{1, \dots, m\}$, x^i is a smooth function $U \to \mathbb{R}$.
Its differential at $p \in M$ is a linear map $(\mathrm{d}x^i)_{|_p} \colon \mathrm{T}_p M \to \mathrm{T}_{x^i(p)} \mathbb{R} \approx \mathbb{R}$.
Each $(\mathrm{d}x^i)_{|_p}$ is an element of the dual vector space $\mathrm{T}_p^* M$ (i.e. $(\mathrm{d}x^i)_{|_p}$ is a ***covector***).

Proposition

Let $p \in U$. There exists a unique basis $(u_1, \dots, u_m)$ of $\mathrm{T}_p M$ such that
$(\mathrm{d}x^i)_{|_p}(u_j) = \delta^i_j$ *where we denote* $\delta^i_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$ *(Kronecker delta).*

Proof. $(u_1, \dots, u_m)$ is the basis of $\mathrm{T}_p M$ whose dual basis is $\big((\mathrm{d}x^1)_{|_p}, \dots, (\mathrm{d}x^m)_{|_p}\big)$.

Notation. u_i is denoted $\frac{\partial}{\partial x^i}$ and called ***coordinate vector***.
For $f \colon U \to \mathbb{R}$, we can write $\frac{\partial f}{\partial x^i}$ instead of $\mathrm{d}f(\frac{\partial}{\partial x^i})$. In particular, we have

$$\frac{\partial x^i}{\partial x^j} = \delta^i_j .$$

Writing vectors and covectors in local coordinates:

Corollary

- *Any tangent vector* $v \in \mathrm{T}_p M$ *is written* $v = \sum_{i=1}^{m} v^i \frac{\partial}{\partial x^i}$ *where* $v^i = (\mathrm{d}x^i)_{|_p}(v)$.
- *Any* ***covector*** $\alpha \in \mathrm{T}_p^* M$ *is written* $\alpha = \sum_{i=1}^{m} \alpha_i \, \mathrm{d}x^i$ *where* $\alpha_i = \alpha(\frac{\partial}{\partial x^i})$.
- *For any smooth function* $f : U \to \mathbb{R}$, *we have* $\mathrm{d}f = \sum_{i=1}^{m} \frac{\partial f}{\partial x^i} \, \mathrm{d}x^i$.

Writing differentials in local coordinates:

Let $f : M \longrightarrow N$ be a smooth map.

Let $(x^i)_{1 \leqslant i \leqslant m}$ be local coordinates on M and $(y^j)_{1 \leqslant i \leqslant n}$ on N.

The ***components*** of f are the functions $f^j : U \longrightarrow \mathbb{R}$ defined by $f^j = y^j \circ f$.

For each $j \in \{1, \ldots, n\}$, we have $\mathrm{d}f^j = \displaystyle\sum_{i=1}^{m} \frac{\partial f^j}{\partial x^i} \, \mathrm{d}x^i$. *At any* $p \in U$*!*

The matrix $\left[\frac{\partial f^j}{\partial x^i} \right]_{\substack{1 \leqslant i \leqslant m \\ 1 \leqslant j \leqslant n}}$ is called the ***Jacobian matrix*** of f. *Depends on* $p \in U$*!*

Proposition

The Jacobian matrix of f at p is the matrix of the linear map $(\mathrm{d}f)|_p : \mathrm{T}_p M \longrightarrow \mathrm{T}_{f(p)} N$ in the bases $\left(\frac{\partial}{\partial x^i} \right)$ of $\mathrm{T}_p M$ and $\left(\frac{\partial}{\partial y^j} \right)$ of $\mathrm{T}_{f(p)} N$.

Proof. See Exercise Sheet #3.

Chapter 5 Smooth maps

5.1 Recap

What do we know about maps between smooth manifolds so far?

- What it means for a map $f\colon M \to N$ to be smooth, or a diffeomorphism.
- The differential $\mathrm{d}f$ of a smooth map.
- How to consider f and compute $\mathrm{d}f$ in charts / local coordinates.

Recall that given charts $(U \ni p, \varphi)$ and $(V \ni f(p), \psi)$, we can look at the map $\hat{f} = \psi \circ f \circ \varphi^{-1}$ instead of f.

$\hat{f}$ is a smooth map $\mathbb{R}^m \to \mathbb{R}^n$ (well-defined on $\varphi(U)$ provided U is small enough).

Let us review some important facts about smooth maps $\mathbb{R}^m \to \mathbb{R}^n$.

Theorem (Inverse inversion theorem)

Let $f : U \subseteq \mathbb{R}^m \to \mathbb{R}^m$ be a smooth map. Let $x_0 \in U$ s.t. If $\mathrm{d}f_{|x_0}$ is invertible.
Then $\exists V \ni x_0$ s.t. $f_V : V \to f(V)$ is a smooth diffeomorphism.

Remark.

- Proof: Banach fixed point theorem for contracting maps between Banach spaces.
- Theorem holds in most other regularity classes C.

Theorem (Implicit function theorem)

Let $f : U \subseteq \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}^m$ be a smooth map.
Let $(x_0, y_0) \in U$ s.t. the partial differential $\mathrm{d}_1 f_{|(x_0,y_0)} : \mathbb{R}^m \to \mathbb{R}^m$ is invertible.
Then $\exists V \ni (x_0, y_0)$ and a smooth map $\varphi : V \cap \mathbb{R}^m \to \mathbb{R}^n$ s.t., for all $(x, y) \in V$:

$$f(x, y) = 0 \Leftrightarrow y = \varphi(x)$$

Rank of a smooth map and the constant rank theorem:

Let $f\colon U \subseteq \mathbb{R}^m \to \mathbb{R}^n$ be a smooth map.

By definition, the ***rank of*** f at $x_0 \in U$ is the rank of the linear map $\mathrm{d}f_{|_{x_0}} : \mathbb{R}^m \to \mathbb{R}^n$.

Theorem (Constant rank theorem)

Let $f\colon U \subseteq \mathbb{R}^m \to \mathbb{R}^n$ be a smooth map. Assume f has constant rank near $x_0 \in U$. There exists diffeos $\varphi\colon U_1 \ni x_0 \to \varphi(U_1) \subseteq \mathbb{R}^m$ and $\psi\colon V \ni f(x_0) \to \psi(V) \subseteq \mathbb{R}^n$ s.t.

$$\psi \circ f \circ \varphi^{-1} = \mathrm{d}f_{|_{x_0}} .$$

Second version:

$$\psi \circ f \circ \varphi^{-1} = (x_1, \ldots, x_m) \mapsto (x_1, \ldots, x_r, 0, \ldots, 0)$$

where r is the rank of f near p.