# Lecture 2 - Complement

Why this complement?

- 20 minutes deficit
- Finish Chapter 1
- Technical section
- Try out slides

## 1.5 Paracompactness and partitions of unity

Compactness properties of topological manifolds:

Let $M$ be a topological manifold, with or without boundary.

(i) $M$ is Hausdorff and locally compact.

(ii) $M$ admits an exhaustion by compact sets.
$M \subseteq \bigcup_{n \in N} K_n$ *with* $K_n \subseteq \operatorname{int} K_{n+1}$.
(In part. $M$ is $\sigma$-compact.)

(iii) $M$ is paracompact.
*Every open cover has a locally finite refinement.*

*Proof: Exercise or [Lee, Top. Manifolds, Thm 4.77]*

# Remark: Topological dimension

## Definition (Top. dimension)

The ***topological dimension*** of $X$ is the smallest integer $m$ such that any open cover of $X$ admits a refinement such that the intersection of $m+1$ subsets of the refinement is always empty.

## Theorem (Top. dimension of manifolds)

*Any topological manifold of dimension $m$ has topological dimension $m$.*

*Proof: Not easy!*

# Normal spaces

**Definition (Normal space)**

A top. space $X$ is called ***normal*** if any pair of disjoint closed sets are contained in disjoint open sets.

**Proposition (See Lee, Thm. 4.81)**

*Any paracompact Hausdorff space is normal.*

**Theorem (Urysohn's lemma (See Lee, Thm. 4.82))**

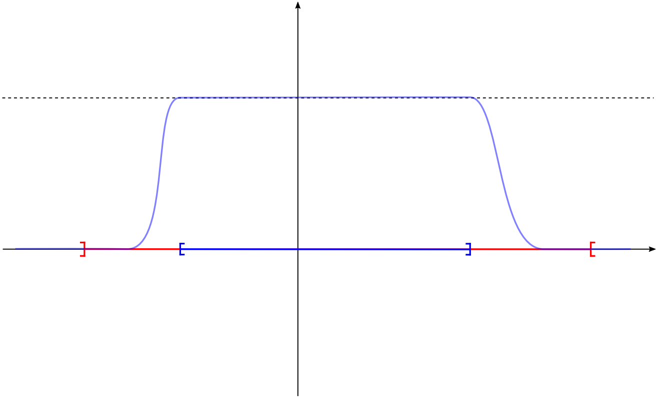
*Let $X$ be normal. For any two disjoint closed sets $A, B \subseteq X$, there exists $f : X \to [0, 1]$ continuous such that $f = 1$ on $A$ and $f = 0$ on $B$.*

# Bump functions

### Corollary (Existence of bump functions)

*Let $X$ be normal. For any $A$ closed $\subseteq U$ open, there exits $f : X \to [0, 1]$ continuous such that $f = 1$ on $A$ and $\operatorname{supp} f \subseteq U$.*

$f$ is called a ***bump function*** for $A$ supported in $U$.



# Partitions of unity

## Definition (Partition of unity)

Let $X$ be a top. space. A ***partition of unity*** is a family of continuous functions $\rho_i \colon X \to [0, 1]$ such that:

(i) $\forall x \in X, \exists U \ni x$ s.t. $\rho_i = 0$ for all but finitely many $i \in I$.

(ii) $\sum_{i \in I} \rho_i = 1$.

It is ***subordinate*** to an open cover $(U_i)_{i \in I}$ if $\operatorname{supp} \rho_i \subseteq U_i$ for all $i \in I$.

## Theorem (Existence of subordinate partitions of unity)

*If $X$ is Hausdorff and paracompact, there exists a partition of unity subordinate to any open cover.*

*Proof: Exercise or [Lee, Thm 4.85].*

# Whitney's embedding theorem

Easy version of Whitney's theorem for topological manifolds:

## Theorem (Whitney's embedding theorem)

*Let $M$ be a compact topological manifold. There exists a topological embedding $F \colon \mathbb{R}^N$ for some $N \in \mathbb{N}$.*

*Proof:*

*(i) By compactness, $M$ is covered by a finite atlas $(U_i, \varphi_i)_{i \in \{1,\ldots,p\}}$.*

*(ii) Let $(\rho_i)_{i \in \{1,\ldots,p\}}$ be a partition of unity subordinate to this cover.*

*(iii) Define $F_i \colon M \to \mathbb{R}^m$ by $F_i(x) = \rho_i(x)\varphi_i(x)\mathbb{1}_{U_i}(x)$ and define*

$$\begin{aligned} F \colon M &\to (\mathbb{R}^m)^p \times \mathbb{R}^p \\ x &\mapsto (F_1(x), \ldots, F_p(x), \rho_1(x), \ldots, \rho_p(x)) \,. \end{aligned}$$

*(iv) $F$ is injective and continuous, therefore it is an embedding by compactness of $M$.*

### Remark (The noncompact case)

As a consequence of topological dimension, any (possibly noncompact) manifold has a *finite* atlas. The previous theorem generalizes to any manifold.

### Remark (Dimension of the embedding)

$M$ always embedds in some $\mathbb{R}^N$, but what is the minimum $N$?