MANIFOLDS
SS 18

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Contents

References

a) Agricola/Friedrich [AF]: Global analysis, AMS 2002; German version (2nd ed.): Vektoranalysis 2011
(The book strives to be more elementary. Nevertheless it covers interesting applications in mathematics and physics. Definitions are only given for submanifolds, but the book is written such that most arguments apply to the general manifold setting.)

b) Lee [Lee]: Introduction to smooth manifolds, Springer 2003
(comprehensive, explicit)

c) Spivak [Spi]: Differential Geometry, vol 1, Publish or Perish 1979
(explicit, geometric, and sharing my point of view)

d) Bredon [Br]: Topology and geometry, Springer 1993
(very concise, with a focus on advanced further topics)

e) Warner [W]: Foundations of differentiable manifolds and Lie groups, Springer 1983
(concise)

f) Boothby [Bo]: An introduction to differentiable manifolds and Riemannian geometry, Academic Press 1975
(an older classic, also explicit)

References for specific topics:

a) Arnold [A]: Mathematical methods of classical mechanics, Springer 1978

b) Guillemin/Pollack [GP]: Differential Topology, Prentice Hall 1974

c) Munkres [M]: Topology, 2nd ed., Prentice Hall 2000

Introduction

This class is aimed at students in their third Bachelor year or in the Master programme. It is taught in English.

The class comprises three parts, namely the introduction to differentiable manifolds, vector fields, and differential forms. For each part, my goal has been to supplement the formal exposition with a concrete result.

The first such result, illustrating the idea of manifolds, is the Whitney embedding theorem. It says that any given abstract manifold can be realized as a submanifold of some Euclidean space.

In the second part, about vector fields, the relevant result is the Frobenius integrability theorem. In the simplest case, it states a condition for a distribution of planes, to admit a surface tangent to the distribution, thereby explaining the geometric meaning of the Lie bracket.

For differential forms, clearly the main result is Stokes' theorem. This theorem generalizes the fundamental theorem of calculus to a form which includes all classical integral theorems, for instance the divergence theorem. I regret that interesting applications cannot be presented appropriately in a class which meets for 90 minutes a week. Also I had to skip some prepared material in class, which here appears in small print.

Problems were presented in the seven problem sessions. The collected material from all previous classes is appended. For 2018 Arthur Windemuth is the author of many new problems, he also wrote up solutions which are included in the present version.

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Darmstadt, July 2018, Karsten Brauckmann

Part 1. Differentiable manifolds and the Whitney embedding theorem

1. Manifolds and differentiability

Differentiable manifolds are the abstract generalization of the notion of submanifolds which are in turn generalizations of curves and surfaces.

Recall that a *submanifold of dimension* n is a subset $M \subset \mathbb{R}^{n+k}$ which in the neighbourhood of $U \subset M$ of each point can be described implicitely, as the zero set of a function defined on ambient space. Another description is parametric, as the image of an embedding of a domain of Euclidean n-space into $\mathbb{R}^{n+k}$; a particular case of a parametrization is a (local) graph. See Sect. 2.6.

Submanifolds are smooth and locally look like deformed Euclidean space $\mathbb{R}^n$. They neither have self-intersections nor boundary, but may have several connected components. Examples include the spheres $\mathbb{S}^n \subset \mathbb{R}^{n+1}$, quadrics, or matrix groups like $\mathsf{O}(n)$ or $\mathsf{GL}(n) \subset \mathbb{R}^{n^2}$.

The definition of a manifold is motivated by spaces which arise without a given ambient *[umgebend]* space:
• Quotient constructions such as $T^n = \mathbb{R}^n/\mathbb{Z}^n$ or $\mathbb{R}P^n = \mathbb{S}^n/\{\pm\,\mathrm{id}\}$.
• Configuration spaces such as the space of polygons in $\mathbb{R}^2$ (given by n-tuples of points) or the space of immersed disks in $\mathbb{R}^2$ with polygonal boundary.

Historically, Riemann presented the intuition for a manifold in his inaugural lecture *[Habilitationsvortrag]* of 1853, using foundational ideas of Gauss. The formal notion of a manifold goes back to Hermann Weyl: In his book *Die Idee der Riemannschen Fläche* from 1913, he elaborated it for the case of surfaces with a complex structure. The ideas became fundamental for the theory of general relativity, developed at the time. In this example, as in most others, the manifold arises with an additional structure. In fact, there is a zoo of such structures on manifolds, Riemannian manifolds, Lie groups, symplectic manifolds, Kähler manifolds, Poisson manifolds, etc.

1.1. Topological manifolds.

The underlying space of a manifold is as follows:

Definition. A *topological space* $(M, \mathcal{O})$ consists of a set M and a family $\mathcal{O}$ of subsets of M, such that
• arbitrary unions und finite intersections of sets in $\mathcal{O}$ are again in $\mathcal{O}$,
• the empty set and M belong to $\mathcal{O}$.
Sets in $\mathcal{O}$ are called *open sets*, sets A whose complements $M \setminus A$ are in $\mathcal{O}$ are called *closed sets*.

Example. For a metric space (M, d) a subset U is in $\mathcal{O}$ if for each point $p \in U$ there is a distance ball $B_r(p) = \{q \in M : d(p, q) < r(p)\}$ contained in U. Please verify that $(M, \mathcal{O})$ is a topological space.

A map $f : M \to N$ between topological spaces is *continuous* if all open sets $V \subset N$ have preimages $f^{-1}(V) \subset M$ which are open. *Homeomorphisms* are bijective continuous maps with continuous inverse.

We will require further properties. The first one has the flavour of a parametric description; however, unlike for submanifolds, the maps run the other way since the manifold is the given object. The other two ensure that the space is well-behaved.

Definition. (i) A topological space $(M, \mathcal{O})$ is *locally Euclidean of dimension* $n \in \mathbb{N}_0$ if each point of M has a neighbourhood homeomorphic to an open subset of $\mathbb{R}^n$. That is, for all $p \in M$ there exists an open subset $U \subset M$ and a homeomorphism $x : U \to x(U) \subset \mathbb{R}^n$. Then (x, U) or x is called a *chart [Karte]* of M. For convenience we always assume that U is connected.
(ii) The topological space $(M, \mathcal{O})$ is *Hausdorff* if for any pair of points $p \neq q \in M$ there are two open sets $U, V \in \mathcal{O}$ with $p \in U$ and $q \in V$ which are disjoint, $U \cap V = \emptyset$.
(iii) The space $(M, \mathcal{O})$ is *second countable [zweit-abzählbar]* if there is a countable base for the topology $\mathcal{O}$. Here, a *base* $\mathcal{B} \subset \mathcal{O}$ is a family of sets such that each open set $U \in \mathcal{O}$ is a union of sets in $\mathcal{B}$.

Example. $\mathbb{R}^n$ is second countable: For $\mathcal{B}$ we can take the balls of rational radius centered at points with rational coordinates.

Definition. A *topological manifold [Mannigfaltigkeit] of dimension* $n \in \mathbb{N}_0$ is a topological space $(M, \mathcal{O})$ which is locally Euclidean of dimension n, Hausdorff, and second countable.

We will write $n = \dim M$. To indicate the dimension, we occasionally write M^n for M.

Examples. 1. $n = 0$: Finite or countable unions of points, such that each point is open.
2. $n = 1$: All connected manifolds are homeomorphic to either $\mathbb{R}$ or $\mathbb{S}^1$. See Guillemin/Pollack, appendix, for a proof. Note that closed intervals are not manifolds.
3. $n = 2$: Each connected compact manifold is homeomorphic to a surface, classified by orientability and genus.
4. Graphs of continuous functions over open sets, for instance a single cone in $\mathbb{R}^3$ (or $\mathbb{R}^n$). However, a double cone in $\mathbb{R}^3$ is not a manifold since it is not locally Euclidean at 0.

Remarks. 1. In the topological setting it is a nontrivial fact that $n \in \mathbb{N}_0$ is uniquely determined: Given a good notion of topological dimension (see, for instance § 50 of Munkres), homeomorphisms preserve dimension. The proof requires tools from algebraic topology. Let us give the simple proof for dimension 1. Consider a homeomorphism f from an open interval I to an open connected set $U \subset \mathbb{R}^n$, where we assume $n \geq 2$. Then, for any $p \in I$, the set $I \setminus \{p\}$ is not connected. But for a homeomorphism, $f(I \setminus \{p\}) = f(I) \setminus \{f(p)\} = U \setminus \{f(p)\}$ is still connected; this contradiction implies $n = 1$. Soon, we will assume differentiability, and then it is straightforward to show that diffeomorphisms preserve dimension, see Thm. 7. So we will not bother about the problem.

2. Metric spaces are always Hausdorff. If they are locally Euclidean and have countably many connected components they are also second countable. Thus the essential remaining property to verify is (i).

3. Suppose M is a Hausdorff space which is locally Euclidean. The second countability of M is equivalent to any of the following reasonable properties:
a) Countably many compact sets cover M.
b) M has countably many connected components; and M is *paracompact*, that is, each open cover has a locally finite subcover. This will ensure a partition of unity exists.
c) M admits a compact exhaustion.

The following terminology is useful:

Definition. (i) If (x, U), (y, V) are two charts of a topological manifold then the map

$$y \circ x^{-1} : \quad x(U \cap V) \to y(U \cap V), \tag{1}$$

is called a *transition map [Kartenwechsel]*.
(ii) An *atlas* of M is a set of charts $\mathcal{A} = \{(x_\alpha, U_\alpha) : \alpha \in A\}$ with $\bigcup_{\alpha \in A} U_\alpha = M$.

Note that by definition a transition map is a continuous, in fact a homeomorphism.

1.2. **Differentiable manifolds.** We want to transfer the notion of differentiability from the parameterizing sets in Euclidean space to manifolds. In order for different charts to lead to a consistent notion, we require:

Definition. (i) We say two charts (x, U), (y, V) are *differentiably compatible [differenzierbar verträglich]* if (1) is a C^∞-diffeomorphism.
(ii) An atlas $\mathcal{A}$ is a *differentiable atlas* if all pairs of charts in $\mathcal{A}$ are differentiably compatible.

Recall that a diffeomorphism is a differentiable map with a differentiable inverse; instead of C^∞, it is usually sufficient to require only C^1 or C^2. Note that the requirement (i) becomes vacuous for the case $U \cap V = \emptyset$.

An atlas is an arbitrary means to describe a manifold. In order to say that two differentiable manifolds agree ("are diffeomorphic") we need to compare homeomorphic manifolds with two perhaps different differentiable atlases. A possible approach is to introduce an equivalence relation on the set of differentiable atlases, and to consider equivalence classes (see, e.g., [GHL]). We follow another approach which avoids the need to deal with classes instead of atlases, and which consists of completing an atlas with all differentiably compatible charts.

Definition. A *differentiable structure* on a topological manifold M with differentiable atlas $\mathcal{A}$ is a set of charts $\mathcal{S} \supset \mathcal{A}$ containing exactly the charts that are differentiably compatible to all charts of $\mathcal{A}$.

Examples. 1. If $\mathcal{A} = \{(\mathrm{id}, \mathbb{R}^n)\}$ then

$$\mathcal{S} = \big\{(f, U) : U \subset \mathbb{R}^n, f\colon U \to \mathbb{R}^n \text{ diffeomorphism onto its image}\big\}.$$

2. We can think of a differentiable structure as a notion telling us which subsets are smooth and which ones have (nonsmooth) corners or edges. Consider, for instance, the following two differentiable structures on $\mathbb{R}^n$: $\mathcal{A} := \{(\mathrm{id}, \mathbb{R}^n)\}$, $\mathcal{B} := \{(f, \mathbb{R}^n)\}$, where f is 1-homogeneous, preserves rays through the origin as sets, and maps the unit cube onto the unit ball. In $\mathcal{B}$ a unit cube (centered at the origin) is a nice differentiable object, while a unit sphere is not (see problems).

The charts added to an atlas by a differentiable structure are differentiably compatible among themselves:

Proposition 1. *A differentiable structure $\mathcal{S}$ is itself a differentiable atlas.*

Hence there is a unique maximal atlas $\mathcal{S}$ containing $\mathcal{A}$, given by all charts which are differentiably compatible with all charts of $\mathcal{A}$. In this sense, a differential structure $\mathcal{S}$ is a *maximal differentiable atlas*.

Proof. Let $\mathcal{A}$ be an atlas and $(x, U), (y, V) \in \mathcal{S}$. We need to show x is differentiably compatible with y. For each point $p \in U \cap V$ there exists a chart $(x_\alpha, U_\alpha) \in \mathcal{A}$ containing p. Then at p we write

$$y \circ x^{-1} = (y \circ x_\alpha^{-1}) \circ (x_\alpha \circ x^{-1}),$$

and each parenthesis is differentiable due to the definition of $\mathcal{S}$, so that the composition is differentiable by the chain rule. $\square$

Definition. A (*differentiable*) *manifold* is a pair $(M, \mathcal{S})$, where M is a topological manifold and $\mathcal{S}$ a differentiable structure. If $\mathcal{A} \subset \mathcal{S}$ is an atlas we will also call $(M, \mathcal{A})$ or M a manifold and say *chart* for a chart of $\mathcal{A}$.

Whenever we say *differentiable* we mean smooth or C^∞. We could define similarly C^k-manifolds or analytic (C^ω) manifolds, by requiring the transition maps are in these classes. Replacing differentiability by holomorphicity gives the notion of a *complex manifold*.

1.3. **Examples of differentiable manifolds.** 1. $\mathbb{R}^n$ is a differentiable manifold with the atlas $\{(\mathrm{id}, \mathbb{R}^n)\}$. We use this differentiable structure on $\mathbb{R}^n$ unless stated otherwise. This means we can and will ignore charts for $\mathbb{R}^n$.

2. The structures $\{(\mathrm{id}, \mathbb{R})\}$ and $\{(x^3, \mathbb{R})\}$ are different (see problems). Similarly, any homeomorphism of $\mathbb{R}^n$ which is not a diffeomorphism gives rise to distinct differentiable structures on $\mathbb{R}^n$.

3. *Spheres* $\mathbb{S}^n := \{p \in \mathbb{R}^{n+1} : p_1^2 + \ldots + p_{n+1}^2 = 1\}$ with $n \in \mathbb{N}$. We use *stereographic projection* onto the equatorial plane to define two charts $x_\pm$. Let $N := (0, \ldots, 0, 1)$ be the north pole and $-N$ the south pole. Given $p \in U_\pm := \mathbb{S}^n \setminus \{\pm N\}$, we determine maps $x_\pm \colon U_\pm \to \mathbb{R}^n$ by requiring that the three points $\pm N$, p, $(x_\pm(p), 0)$ are on a line. That is, there is $\lambda \neq 1$ with

$$\big(x_\pm(p), 0\big) \overset{!}{=} \lambda p \pm (1 - \lambda) N.$$

The first n coordinates of this equation give

$$x_\pm(p) = \lambda(p_1, \ldots, p_n),$$

while the last coordinate determines λ:

$$0 = \lambda p_{n+1} \pm (1 - \lambda) = \pm 1 + \lambda(\mp 1 + p_{n+1}) \quad \Rightarrow \quad \lambda = \frac{\mp 1}{\mp 1 + p_{n+1}} = \frac{1}{1 \mp p_{n+1}}$$

Thus our charts are

$$x_\pm \colon U_\pm := \mathbb{S}^n \setminus N_\pm \to \mathbb{R}^n, \qquad x_\pm(p) := \frac{1}{1 \mp p_{n+1}}(p_1, \ldots, p_n).$$

We claim that $\mathcal{A} := \big\{(x_+, U_+), (x_-, U_-)\big\}$ is an atlas.
• Clearly, $U_+ \cup U_- = \mathbb{S}^n$.
• The charts are bijective: Indeed,

$$x_\pm^{-1} \colon \mathbb{R}^n \to U_\pm, \quad x_\pm^{-1}(u) := \frac{1}{|u|^2 + 1}\Big(2u,\ \pm(|u|^2 - 1)\Big)$$

are inverses to $x_\pm$ since for all $u \in \mathbb{R}^n$

$$\begin{aligned} x_\pm\big(x_\pm^{-1}(u)\big) &= x_\pm\left(\frac{2u}{|u|^2+1},\ \pm(1 - \frac{2}{|u|^2+1})\right) \\ &= \frac{1}{1 - \big(1 - \frac{2}{|u|^2+1}\big)} \cdot \frac{2u}{|u|^2+1} = \frac{1}{\frac{2}{|u|^2+1}} \cdot \frac{2u}{|u|^2+1} = u. \end{aligned}$$

and similarly so $\big(x_\pm^{-1} \circ x_\pm\big)(p) = p$ (check!).
• The charts and their inverses are continuous with respect to the relative topology: The coordinate functions are continuous functions, and so is their insertion into continuous

functions.

• Finally, the two transition maps, $x_\pm \circ x_\mp^{-1}$ which map $x_\mp(U_+ \cap U_-) = \mathbb{R}^n \setminus \{0\}$ into itself,

$$(2)\qquad \begin{aligned}\big(x_\pm \circ x_\mp^{-1}\big)(u) &= x_\pm\left(\frac{2u}{|u|^2+1},\ \mp(1-\frac{2}{|u|^2+1})\right)\\ &= \frac{1}{1+(1-\frac{2}{|u|^2+1})}\cdot\frac{2u}{|u|^2+1} = \frac{1}{2-\frac{2}{|u|^2+1}}\cdot\frac{2u}{|u|^2+1} = \frac{u}{|u|^2},\end{aligned}$$

are differentiable. Geometrically they represent an inversion in the equatorial unit sphere $\mathbb{S}^{n-1} \subset \mathbb{R}^n$.

A simpler choice of atlas for $\mathbb{S}^n$ uses the $2(n+1)$ hemispheres $H_\pm^k := \{p \in \mathbb{S}^n : \langle p, \pm e_k\rangle > 0\}$, which also cover $\mathbb{S}^n$. The charts are given by projection onto the hyperplane $\{x_k = 0\}$, the inverses are the graph representations of hemispheres.

Stereographic projection is not only nicer in that only two charts are sufficient, but it has a useful additional property: It is conformal, that is, angle preserving. The same property also holds for the inversion map.

4. *Projective spaces* $\mathbb{K}P^n$ where $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$; here $\mathbb{H}$ denotes quaternions. These spaces are the sets of one-dimensional $\mathbb{K}$-subspaces ("lines") of $\mathbb{K}^{n+1}$.

The relation $u \sim \lambda u$ for $\lambda \in \mathbb{K}$ is an equivalence relation on $\mathbb{K}^{n+1}\setminus\{0\}$. Its classes, endowed with the quotient topology, form a topological space

$$\mathbb{K}P^n := \big\{[u] : u = (u_1, \dots, u_{n+1}) \in \mathbb{K}^{n+1} \setminus \{0\}\big\}$$

of dimension n, $2n$, or $4n$, depending on the choice of $\mathbb{K}$.

To introduce a differentiable structure we use *homogeneous coordinates*. These are the homeomorphisms

$$x_i \colon U_i := \big\{[u] : u_i \neq 0\big\} \subset \mathbb{K}P^n \to \mathbb{K}^n, \qquad x_i\big([u]\big) := \frac{1}{u_i}(u_1, \dots, \widehat{u_i}, \dots, u_{n+1})$$

for $i = 1, \dots, n+1$; the hat $\widehat{\cdot}$ indicates an entry to be omitted, and the target space $\mathbb{K}^n$ is either $\mathbb{R}^n$, $\mathbb{R}^{2n}$, or $\mathbb{R}^{4n}$. Represent $[u]$ by u and λu to see that x_i is well-defined. Observe also that x_i is the identity for the representative u of $[u]$ contained in the affine hyperplane $H_i := \{u \in \mathbb{K}^{n+1} : u_i = 1\}$. Therefore it is clear that the inverse is given by

$$x_i^{-1} \colon \mathbb{K}^n \to U_i \qquad x_i^{-1}(u_1, \dots, u_n) := [u_1, \dots, u_{i-1}, 1, u_i, \dots, u_n].$$

In particular, x_i is a homeomorphism. We also confirm by calculation that x_i^{-1} is the inverse of x_i:

$$(x_i \circ x_i^{-1})(u) = x_i\big([u_1, \dots, u_{i-1}, 1, u_i, \dots, u_n]\big) = \frac{1}{1}(u_1, \dots, u_{i-1}, u_i, \dots, u_n) = u \quad \forall u \in \mathbb{K}^n,$$

$$(x_i^{-1} \circ x_i)([u]) = x_i^{-1}\Big(\frac{1}{u_i}(u_1, \dots, \widehat{u_i}, \dots, u_{n+1})\Big) = \Big[\frac{u_1}{u_i}, \dots, 1, \dots, \frac{u_{n+1}}{u_i}\Big] = [u] \qquad \forall u \in U_i.$$

We claim that the collection of our charts, $\mathcal{A} := \{(x_i, U_i) : i = 1, \ldots, n+1\}$, forms an atlas. Clearly, the U_i cover $\mathbb{K}P^n$. Moreover, $\mathcal{A}$ is differentiable, since for $j < i$ and all $u \in x_i\big(U_i \cap U_j\big) = \{u \in \mathbb{K}^n : u_i \neq 0 \text{ and } u_j \neq 0\}$

$$\big(x_j \circ x_i^{-1}\big)(u) = x_j\big([u_1, \ldots, u_{i-1}, 1, u_i, \ldots, u_n]\big) = \frac{1}{u_j}(u_1, \ldots, \widehat{u_j}, \ldots u_{i-1}, 1, u_i, \ldots, u_n).$$

Similarly for $j > i$. This proves differentiability of the transition maps.

5. *Grassmannians [Grassmann-Räume]* $G(k, n)$ are the sets of k-dimensional subspaces of $\mathbb{R}^n$. Taking orthogonal complements we see that $G(k, n) = G(n-k, n)$. The case $k = 1$ is real projective space, $\mathbb{R}P^{n-1} = G(1, n) = G(n-1, n)$. These spaces are easy to describe as quotient spaces, but explicit coordinates are somewhat tedious.

6. *Lie groups* are manifolds which are groups, with a continuous group operation. Typical examples are $\mathsf{GL}(n, \mathbb{R})$, $\mathsf{O}(n, \mathbb{R})$, $\mathsf{SL}(n, \mathbb{R})$. The sign of the determinant indicates that the first two matrix groups are not connected; they can be shown to have exactly two connected components. Also the tori $T^n = \mathbb{R}^n/\mathbb{Z}^n$ are Lie groups. The only spheres which are Lie groups are $\mathbb{S}^1$ and $\mathbb{S}^3$; the group structure for the latter is given by the unit quaternions.

7. The most important construction of manifolds is in terms of quotients. We avoid to go into the necessary technicalities here. If the quotient is taken in terms of a discrete group action, then the dimension is preserved; examples are $T^n = \mathbb{R}^n/\mathbb{Z}^n$, or $\mathbb{R}P^n = \mathbb{S}^n/\{\pm\,\mathrm{id}\}$. If the quotient is by a Lie subgroup, the dimension can decrease, for instance $\mathbb{S}^2 = \mathbb{S}^3/\mathbb{S}^1$.

8. Another standard example are n-dimensional submanifolds or $\mathbb{R}^{n+k}$.

9. Each open subset U of a manifold M is a manifold itself. The structure $\mathcal{S}$ is given by those maps (x, V) of the differentiable structure of M, for which V is an open subset of U.

1.4. **Differentiable maps.** We define differentiability of mappings between manifolds by requiring that their composition with charts be differentiable:

Definition. Let M and N be differentiable manifolds and $f\colon M \to N$ be continuous.
(*i*) f is called *differentiable at* $p \in M$ if for some chart (x, U) of M at p and some chart (y, V) of N at $f(p)$ the locally defined map $y \circ f \circ x^{-1}$ is differentiable at $x(p)$.
(*ii*) f is *differentiable* if f is differentiable at all $p \in M$.

Our definition is independent of the particular charts chosen: With respect to other charts $\tilde{x}$ at p and $\tilde{y}$ at $f(p)$ we can write, on suitable domains,

$$\tilde{y} \circ f \circ \tilde{x}^{-1} = (\tilde{y} \circ y^{-1}) \circ (y \circ f \circ x^{-1}) \circ (x \circ \tilde{x}^{-1}).$$

Note that the two transition maps on the right hand side are diffeomorphisms. Hence, by the chain rule, $\tilde{y} \circ f \circ \tilde{x}^{-1}$ is differentiable if and only if $(y \circ f \circ x^{-1})$ is.

By the same token, differentiability is preserved under composition: To see this, write $z \circ f \circ g \circ x^{-1} = (z \circ f \circ y^{-1}) \circ (y \circ g \circ x^{-1})$ and apply the chain rule once again.

Examples. 1. Trivially, the identity on M is differentiable since transition maps are differentiable.
2. Each chart $x_\alpha \colon U_\alpha \to \mathbb{R}^n$ of a differentiable manifold M becomes a differentiable mapping when considered a map of manifolds: indeed, $\operatorname{id} \circ x_\alpha \circ x_\beta^{-1}$ is a transition map hence differentiable.

For instance, stereographic projection $x_\pm$ is a differentiable mapping from the manifold $\mathbb{S}^n \setminus \{\pm N\}$ to $\mathbb{R}^n$.

Definition. A *diffeomorphism* $f \colon M \to N$ between manifolds is a homeomorphism such that f and f^{-1} are differentiable. Then we call M and N *diffeomorphic (manifolds).*

Note that this definition works for any choice of atlas for the manifolds. If M is diffeomorphic to N then $\dim M = \dim N$ (why?). Diffeomorphic manifolds are considered indistinguishable, just like isomorphic vector spaces or homeomorphic topological spaces. As a precaution, let us say that often we refer to a diffeomorphism without specifying its target explicitly; then, what we mean by diffeomorphism is a diffeomorphism onto its image.

Examples. 1. $\mathbb{R}^n$ and $B^n = \{x \in \mathbb{R}^n : |x| < 1\}$ are diffeomorphic via $x \mapsto \frac{x}{|x|} \operatorname{arctanh} |x|$.
2. T^2 and the torus of revolution are diffeomorphic (via?).
3. If (x, U) is a chart of any n-manifold M then $U \subset M$ and $x(U) \subset \mathbb{R}^n$ are diffeomorphic.

2. Tangent space and differentiable maps

A differentiable manifold is distinguished from a topological manifold by having a tangent space.

2.1. **Equivalence classes of curves.** The tangent space to a submanifold $M \subset \mathbb{R}^{n+k}$ can be represented by the set of tangent vectors to curves within M. Similarly, we want to represent the tangent space of a manifold at a point p by the set of tangent vectors of curves through p. We must identify curves which have the same tangent vector in a chart.

Definition. (i) A (differentiable) *curve* c on a manifold M is a differentiable map $c\colon I \to M$, where I is an open interval. We say c is a *curve through* $p \in M$ if $0 \in I$ and $c(0) = p$.
(ii) A *tangent vector* to M^n at $p \in M$ is an equivalence class of curves through p under the following relation:

$$c_1 \sim c_2 \quad :\Longleftrightarrow \; \exists \text{ chart } x \text{ at } p\colon \; (x \circ c_1)'(0) = (x \circ c_2)'(0) \in \mathbb{R}^n$$

The *tangent space of* M *at* $p \in M$ is the set of tangent vectors $T_pM := \{[c]\colon c(0) = p\}$.

That is, for a given point $p = c(0) \in M$, called the *foot point*, a tangent vector $[c] \in T_pM$ is represented by the vector

$$\xi := (x \circ c)'(0) \in \mathbb{R}^n,$$

called the *principal part [Hauptteil]* of $[c]$ with respect to the chart x.

To see the relation $\sim$ is independent of the chart x at p chosen, consider yet another chart y at p. Then, for $i = 1, 2$,

$$(y \circ c_i)'(0) = (y \circ x^{-1} \circ x \circ c_i)'(0) \overset{\text{chain rule}}{=} \underbrace{d(y \circ x^{-1})_{x(p)}}_{\text{independent of } i=1,2} \cdot (x \circ c_i)'(0). \tag{3}$$

We can read (3) to say that principal parts of tangent vectors at p transform with the Jacobian of the transition map:

Theorem 2 (Transformation rule for tangent vectors)**.** *Let $v \in T_pM$ be represented by the principal parts ξ and η with respect to two charts x and y at p, respectively. Then*

$$\eta = d(y \circ x^{-1})_{x(p)}\, \xi. \tag{4}$$

Examples. 1. Let M be the sphere $\mathbb{S}^2$ with stereographic charts $x_\pm$. Consider the longitude $c(t) = (\cos t, 0, \sin t)$ through $p := c(0) = (1, 0, 0)$. With respect to the charts $x_\pm$ let us compute the principal parts:

$$x_\pm \circ c = \left(\frac{\cos t}{1 \mp \sin t}, 0\right) \quad \Rightarrow \quad \frac{d}{dt}(x_\pm \circ c)\Big|_{t=0} = \left(\frac{0 - (\mp 1)}{1}, 0\right) = (\pm 1, 0) =: \xi_\pm \tag{5}$$

The transformation rule states that $\xi_- = d(x_- \circ x_+^{-1})\xi_+$. Indeed, the transition map of the two stereographic projections (2) is inversion in the unit circle, and so its linearisation is a reflection in the y-line tangent to the circle, agreeing with our result.

2. The following calculation is better to digest in form of a sketch. Consider $\mathbb{R}P^1$ with its two charts x_1, x_2. At $p = [(2,1)]$ we have $x_1(p) = 1/2 \in \mathbb{R}$ and $x_2(p) = 2 \in \mathbb{R}$. Consider the curve $c(t) = [(2, 2t+1)] \in \mathbb{R}P^1$ through p. Then $x_1(c(t)) = t + 1/2$ with $\xi_1 = 1$, and $x_2(c(t)) = 2/(2t+1)$ with $\xi_2 = (\frac{1}{t+1/2})'(0) = -4$. Recall $(x_2 \circ x_1^{-1})(u) = 1/u$, and so $d(x_2 \circ x_1^{-1})\big|_{u=1/2} = -1/u^2|_{u=1/2} = -4$. Thus indeed $\xi_2 = d(x_2 \circ x_1^{-1})\big|_{u=1/2}\xi_1$.

A map is linear if it respects the vector space operations of addition and scalar multiplication. In particular, the linear map $d(y \circ x^{-1})_{x(p)}$ has this property, and so we may use (4) to define addition and scalar multiplication of tangent vectors unambiguously by the operations on principal parts:

Theorem 3. *For M an n-dimensional differentiable manifold, the tangent space T_pM is an n-dimensional vector space, and consequently for a given chart x at p the map*

$$(6) \qquad \mathbb{R}^n \to T_pM, \quad \xi \mapsto [c_{x,p,\xi}], \qquad \text{where } c_{x,p,\xi}(t) := x^{-1}\big(x(p) + t\xi\big) \text{ for } t \text{ small,}$$

is an isomorphism.

Note that $x(p)+t\xi$ traces out a straight line in the chart image, with directional derivative $\xi = \frac{d}{dt}(x(p)+t\xi)|_{t=0}$, and that the principal part ξ represents a tangent vector by definition. To rephrase the theorem once again, let us say that $\lambda[c_{x,p,\xi_1}] + [c_{x,p,\xi_2}] := [c_{x,p,\lambda\xi_1+\xi_2}]$ for $\lambda \in \mathbb{R}$ and $\xi_1, \xi_2 \in \mathbb{R}^n$ is well-defined independently of the chart x at p chosen.

The vector space isomorphism (6) maps the coordinate basis $(b_1, \ldots, b_n)$ of $\mathbb{R}^n$ to the *standard basis of T_pM with respect to the chart x*,

$$(e_1, \ldots, e_n)\,(p) := \big([c_{x,p,b_1}],\ \ldots\ , [c_{x,p,b_n}]\big).$$

Each $v \in T_pM$ can then be represented by a linear combination

$$(7) \qquad v = [c_{x,p,\xi}] = \big[c_{x,p,\sum \xi^i b_i}\big] = \sum \xi^i \big[c_{x,p,b_i}\big] = \sum \xi^i e_i(p).$$

However, another chart (y, V) at p will lead to a possibly different standard basis. We say that T_pM does not have a *canonical basis.*

We finally introduce a commonly used matrix notation for the transformation of principle parts. From (4) we have $\eta^j = \sum_{i=1}^n \partial_i (y \circ x^{-1})^j\big|_{x(p)}\, \xi^i$. It is common to write

$$\frac{\partial y^j}{\partial x^i}(p) := \partial_i\big(y \circ x^{-1}\big)^j(x(p)) \quad \text{for } i, j = 1, \ldots, n,$$

so that the transformation rule takes the appearance of the chain rule in Euclidean space, $\eta^j = \sum_{i=1}^n \frac{\partial y^j}{\partial x^i}(p)\, \xi^i$.

Examples. 1. For $\mathbb{R}P^2$ and the point $p = [(1,1,1)]$, the chart x_1 gives the standard basis $[c_2(t) = (1, 1+t, 1)]$, $[c_3(t) = (1, 1, 1+t)]$, while x_2 gives the standard basis $[c_1(t) = (1+t, 1, 1)]$, $[c_2(t)]$.

1. Again we consider $p := (1,0,0) \in M := \mathbb{S}^2$. For x_+, the standard basis at p can be represented by the curves $c_1(t) = (\cos t, 0, \sin t)$ and $c_2(t) = (\cos t, \sin t, 0)$, while for x_- the standard basis at p is $-c_1(t)$ and $c_2(t)$; that is, the first tangent vectors are opposite. Indeed, the first tangent vector was computed in (5), while the second is immediate since

both charts map the equatorial unit circle of $\mathbb{S}^2$ to the unit circle in the plane, that is, $x_\pm(\cos t, \sin t, 0) = (\cos t, \sin t)$.

3. On $\mathbb{R}^n$ with its trivial atlas $\{\mathrm{id}, \mathbb{R}^n\}$ the isomorphism (6) is canonical, an identification:

$$[c_{\mathrm{id},p,\xi}] = [t \mapsto p + t\xi] \overset{!}{=} \xi. \tag{8}$$

That is, for $\mathbb{R}^n$ a class of curves is equated with a vector, namely the common tangent! In view of this identification we have

$$T_p\mathbb{R}^n = \mathbb{R}^n. \tag{9}$$

Similarly for open subsets of $\mathbb{R}^n$.

2.2. **The tangent bundle TM.** We want to collect all tangent spaces T_pM in a single space, described in terms of foot points and principal part. This will allow us to decide on the continuity and differentiability of objects such as vector fields.

Definition. The *tangent bundle* TM of a manifold M has the disjoint union

$$TM := \bigsqcup_{p\in M} T_pM$$

as the underlying set. To define charts for TM take an atlas $\mathcal{A}_M = \{(x_\alpha, U_\alpha) : \alpha \in A\}$ of M and let

$$y_\alpha \colon \bigsqcup_{p\in U_\alpha} T_pM \to x_\alpha(U_\alpha) \times \mathbb{R}^n \subset \mathbb{R}^{2n}, \qquad y_\alpha([c]) := \Big(x_\alpha(c(0)),\, (x_\alpha \circ c)'(0)\Big). \tag{10}$$

We define a topology on TM by requiring that these charts are homeomorphisms.

Note that if a chart of TM contains a single tangent vector $v \in T_pM$ then it contains the entire vector space T_pM: the manifold description is local only in the foot point.

Theorem 4. *The charts defined by* (10) *are differentiably compatible and so*

$$\mathcal{A}_{TM} := \Big\{(y_\alpha, \bigsqcup_{p\in U_\alpha} T_pM) : \alpha \in A\Big\}$$

is an atlas of TM. It makes TM into a $2n$-dimensional differentiable manifold with a unique differentiable structure independent of the choice of atlas $\mathcal{A}_M$ for M.

Proof. Clearly $\mathcal{A}_{TM}$ covers TM. We show that two charts of this atlas are differentiably compatible. So suppose $p \in U_\alpha \cap U_\beta$ and $[c] \in T_pM$, where $p = c(0)$. Then

$$\begin{aligned}
\Big(y_\beta \circ y_\alpha^{-1}\Big)\Big(x_\alpha(p),\, (x_\alpha \circ c)'(0)\Big) &= y_\beta\big([c]\big) = \Big(x_\beta(p),\, (x_\beta \circ c)'(0)\Big) \\
&= \Big((x_\beta \circ x_\alpha^{-1} \circ x_\alpha)(p),\;\; (x_\beta \circ x_\alpha^{-1} \circ x_\alpha \circ c)'(0)\Big) \\
&= \Big((x_\beta \circ x_\alpha^{-1})(x_\alpha(p)),\;\; d(x_\beta \circ x_\alpha^{-1})_{x_\alpha(p)}\,(x_\alpha \circ c)'(0)\Big)
\end{aligned}$$

where $x_\beta \circ x_\alpha^{-1}$ is differentiable as a transition map, and its differential $d(x_\beta \circ x_\alpha^{-1})$ is smooth as the differential of a smooth map.

Moreover, our calculation confirms:
• The differentiable structure on TM is independent of the choice of atlas $\mathcal{A}$ of M.
• The charts y_α and y_β induce the same topology from $\mathbb{R}^{2n}$ on TM.
We leave it as an exercise to prove that TM is Hausdorff and second countable. □

Examples. 1. For $M = \mathbb{S}^1$ the set TM is diffeomorphic to the cylinder $\mathbb{S}^1 \times \mathbb{R}$.
2. However, for $M = \mathbb{S}^2$ the tangent bundle cannot be homeomorphic to the product $\mathbb{S}^2 \times \mathbb{R}^2$ since each vector field on $\mathbb{S}^2$ is known to have a zero (problems?).

Remark. The tangent bundle is a particular case of a *vector bundle* $\pi\colon E \to B$ of manifolds E^{n+k} and B^n. By definition, charts of the bundle E are defined in terms of charts (x_α, U_α) of B, which map subsets $E_\alpha \subset E$ to $x_\alpha(U_\alpha) \times \mathbb{R}^k$, whereby commuting with π. For instance the cylinder E is a vector bundle with $k = 1$ over $B = \mathbb{S}^1$. It is trivial, that is, diffeomorphic to a product, $E = B \times \mathbb{R}^k$. The Möbius strip, however, is not trivial, but needs at least two charts.

2.3. **Differential.** Remember that by definition a mapping between manifolds is differentiable if the composition with charts is a differentiable Euclidean map. We can now define its Jacobian:

Definition. Let $f\colon M \to N$ be a differentiable mapping between two manifolds M, N. Its *differential* (or tangent map) is the map $df\colon TM \to TN$, defined by

$$df_p\colon T_pM \to T_{f(p)}N, \qquad df[c] := [f \circ c] \qquad \text{where } c(0) = p.$$

Other notation for df includes f_* (*push-forward*), f', or Tf.

Let us show that df is well-defined. Suppose $v \in T_pM$ is represented by two curves c_1, c_2. Then, for $i = 1, 2$,

$$\frac{d}{dt}(y\circ f \circ c_i)(0) = \frac{d}{dt}\big((y\circ f \circ x^{-1}) \circ (x \circ c_i)\big)(0) = d(y\circ f\circ x^{-1})_{x(p)} \cdot \frac{d}{dt}(x\circ c_i)(0), \tag{11}$$

and so indeed $df[c_1] = [f \circ c_1]$ agrees with $df[c_2] = [f \circ c_2]$.

We now assert properties which are well-known for the Euclidean case.

Theorem 5. (*i*) *For each $p \in M$, the restriction $df_p\colon T_pM \to T_{f(p)}N$ is linear.*
(*ii*) *The differential $df\colon TM \to TN$ is a differentiable map.*

Proof. (*i*) To prove linearity, consider (11), which says that the principal part of $[f \circ c]$ depends linearly on the principal part of $[c]$. But by (6) the space of principal parts is isomorphic to the tangent space, hence linearity is preserved.

(*ii*) Compose df with charts of TM and TN. In the resulting commutative diagram we need to check that the principal part of the image depends on the principal part of the preimage in a differentiable way. We leave this as an exercise. □

Theorem 6. *The chain rule* $d(f \circ g)|_p = df_{g(p)} \circ dg_p$ *holds for differentiable maps between manifolds.*

Proof. This is immediate from

$$d(f \circ g)[c] = \big[(f \circ g) \circ c\big] = \big[f \circ (g \circ c)\big] = df[g \circ c] = df\big(dg[c]\big) = (df \circ dg)[c],$$

taken at the appropriate points. □

Theorem 7. *Let* $f \colon M^m \to N^n$ *be a differentiable map. Suppose* df_p *is an isomorphism of vector spaces. Then*
(*i*) $m = n$, *and*
(*ii*) p *has a neighbourhood* W *such that* $f|_W$ *is a diffeomorphism onto its image.*

Proof. (*i*) This is a linear algebra fact.
(*ii*) Pick charts x at p and y at $f(p)$. Then the inverse mapping theorem *[Umkehrsatz]* proves that $x(p)$ has a neighbourhood $W'\big(x(p)\big) \subset \mathbb{R}^m$ such that $y \circ f \circ x^{-1}$ is a diffeomorphism onto its image. As charts x, y are diffeomorphisms, the set $W := x^{-1}(W')$ satisfies the claim. □

A *local diffeomorphism* is a map $f \colon M \to N$ satisfying statement (*ii*) at each $p \in M$. That is, each $p \in M$ has a neighbourhood W so that $f|_W \colon W \to f(W)$ is a diffeomorphism.

Example. $t \mapsto e^{it}$ is a local, but not a global diffeomorphism between $\mathbb{R}$ and $\mathbb{S}^1 \subset \mathbb{C}$.

2.4. Immersions and embeddings.

Definition. Let $f \colon M \to N$ be a differentiable map between manifolds M and N.
(*i*) f is an *immersion* if its differential $df_p \colon T_pM \to T_{f(p)}N$ is injective for all $p \in M$.
(*ii*) f is an *embedding [Einbettung]*, if $f \colon M \to N$ is an immersion and a homeomorphism onto its image.

For (*ii*), the topology of the subset $Y := f(M) \subset N$ is the *subspace topology*: If $\mathcal{O}_N$ are the open sets of N then $\mathcal{O}_Y := \{U \cap Y : U \in \mathcal{O}_N\}$. It is easy to see that this is a topology.

A differentiable homeomorphism between Euclidean sets whose differential is everywhere invertible is a diffeomorphism. Hence, in situation (*ii*) we can also conclude that f is a diffeomorphism onto its image.

Examples. 1. A curve $c\colon I \to \mathbb{R}^n$ is an immersion provided $dc_t = c'(t) \neq 0$, that is, the curve is *regular*. For instance, the curves $t \mapsto (t^2, t^3)$ or $t \mapsto (t^3, t^3)$ are not immersions.

2. Immersions but not embeddings:
- $e^{it}\colon \mathbb{R} \to \mathbb{C}$.
- A *figure eight [Lemniskate]* $c\colon \mathbb{S}^1 \to \mathbb{R}^2$, $c(e^{it}) = (\sin t, \sin 2t)$.

To define an embedding, it may appear sufficient to require an immersion is injective, but not necessarily a homeomorphism. In the next subsection we will show that indeed an injective immersion of a compact manifold is an embedding. However, injective immersions of non-compact manifolds are not necessarily embeddings, as the topology of preimage and the image can differ:

Examples. 1. Consider an injective curve $c\colon (0,1) \to \mathbb{R}^2$ with a point of contact, e.g. $\lim_{t\to 0} c(t) = c(\frac{1}{2})$. Then the image of small open intervals containing $\frac{1}{2}$ is not open: Any open neighbourhood of $c(\frac{1}{2})$ will contain the image $c((0,\varepsilon))$ for small ε. Therefore, c is not a homeomorphism onto its image.
2. Take a line with irrational slope *[Steigung]* in $\mathbb{R}^2$ and project it into the torus $T^2 = \mathbb{R}^2/\mathbb{Z}^2$. The result is an injective immersion, but not a homeomorphism (argue as in the previous example).

If a linear mapping from $\mathbb{R}^n$ to $\mathbb{R}^\ell$ is injective, then $n \leq \ell$. Thus immersions $f\colon M^n \to N^\ell$ have a *codimension* $k = \ell - n \geq 0$.

Locally, an immersion is an embedding:

Theorem 8. *Let $f\colon M^n \to N^{n+k}$ be an immersion. Then each $p \in M$ has a neighbourhood $U \subset M$ such that $f|_U$ is an embedding.*

Proof. We will find a suitable local extension of f and apply the Inverse Mapping Theorem to prove the embedding property of the extension, and therefore of f.

We choose charts $(x, \tilde{U})$ at p and (y, V) at $f(p)$. Upon shrinking of $\tilde{U}$ if necessary we may assume $f(\tilde{U}) \subset V$. The local representation of f is

$$\varphi := y \circ f \circ x^{-1}\colon x(\tilde{U}) \to y(V), \qquad \varphi(u) = \big(\varphi^1(u),\ \dots,\ \varphi^{n+k}(u)\big),$$

where $u = (u^1, \dots, u^n)$. By assumption, $\operatorname{rank}(d\varphi) = n$ and so the $(n+k) \times n$-Jacobian $\big(\partial_j \varphi^i(u)\big)$ has an $n \times n$-submatrix with rank n. Assuming $x(p) = 0 \in \mathbb{R}^n$ we may renumber our φ-coordinates such that at this point the $n \times n$-matrix $\big(\partial_j \varphi^i(0)\big)_{1 \leq i,j \leq n}$ has rank n.

We extend φ by setting

$$\psi\colon x(\tilde{U}) \times \mathbb{R}^k \to \mathbb{R}^{n+k}, \qquad \psi(u,t) := \varphi(u) + (0,t), \qquad \text{where } t = (t^1, \dots, t^k). \tag{12}$$

Then $\psi(u,0)=\varphi(u)$, and the Jacobian

$$J\psi = \begin{pmatrix} (\partial_j\varphi^i)_{1\le i,j\le n} & 0 \\ (\partial_j\varphi^{n+i})_{1\le i\le k,\,1\le j\le n} & 1_k \end{pmatrix}$$

has rank $n+k$ at the point $(u,t)=(0,0)$, due to determinant development.

By the Inverse Mapping Theorem there exists a neighbourhood $\Omega\subset x(\tilde U)\times\mathbb{R}^k$ of $0\in\mathbb{R}^{n+k}$, such that ψ maps Ω diffeomorphically onto $\psi(\Omega)\subset\mathbb{R}^{n+k}$. Define Ω_0 as the slice

$$\Omega_0\times\{0\} := \Omega\cap\big(\mathbb{R}^n\times\{0\}\big).$$

The restriction of a homeomorphism is a homeomorphism onto its image. Hence $\varphi|_{\Omega_0} = \psi|_{\Omega_0\times\{0\}}$ is a homeomorphism of Ω_0 onto its image in $\mathbb{R}^{n+k}$. Moreover charts are homeomorphisms, and so the restriction of $f=y^{-1}\circ\varphi\circ x$ to $U:=x^{-1}(\Omega_0)$ is a homeomorphism onto its image, hence an embedding. $\square$

An immersion can be assigned a standard form by choosing adapted charts:

Corollary 9. *If $f\colon M^n\to N^{n+k}$ is an immersion then for each $p\in M$ there exists a chart (x,U) of M at p and a chart $(\tilde y,\tilde V)$ of N at $f(p)$, such that for all $u=(u_1,\dots,u_n)\in x(U)$*

$$\big(\tilde y\circ f\circ x^{-1}\big)(u)=(u,0)\ \in\mathbb{R}^{n+k}.$$

Moreover, $\tilde y(f(U))=\tilde y(\tilde V)\cap\big(\mathbb{R}^n\times\{0\}\big)$.

Proof. The maps y and ψ from the previous proof are diffeomorphisms onto their images. Hence

$$\tilde y:=\psi^{-1}\circ y\colon\ \ \tilde V:=y^{-1}\big(\psi(\Omega)\big)\to\Omega$$

is also a diffeomorphism, and so defines a chart of N compatible with the differentiable structure. With respect to $\tilde y$ the immersion f has the following local representation:

$$\tilde y\circ f\circ x^{-1}=(\psi^{-1}\circ y)\circ f\circ x^{-1}=\psi^{-1}\circ\varphi\colon\quad \Omega_0\to\Omega.$$

Since $\psi(u,0)=\varphi(u)$ we have $(\psi^{-1}\circ\varphi)(u)=(u,0)$, as desired. Consequently,

$$\tilde y(f(U))=(\tilde y\circ f\circ x^{-1})(\Omega_0)=\Omega_0\times\{0\}=\Omega\cap(\mathbb{R}^n\times\{0\})=\tilde y(\tilde V)\cap(\mathbb{R}^n\times\{0\}).$$ $\square$

2.5. **Some topology.** Our goal is the assertion:

Proposition. *Suppose M is a compact topological space and N is Hausdorff. Then an injective continuous map $f\colon M\to N$ is a homeomorphism onto its image. Consequently, any injective immersion $f\colon M\to N$ of a compact manifold M is an embedding.*

The closedness of f is the key to proving this property. A map $f\colon X\to Y$ between topological spaces is a *closed map* if each closed subset $A\subset X$ has a closed image $f(A)\subset Y$.

Example. Recall the first Example under 3. from p. 14: It gives an immersion c of the open interval into $\mathbb{R}^2$ (with "touching point") which is not closed: The (relatively) closed set $(0,\varepsilon]$ is mapped to a set in $\mathbb{R}^2$ which is not closed since it fails to contain the limit of the sequence $c(1/n)$.

Lemma. *If $f\colon X \to Y$ is a continuous map of topological spaces, where X is compact and Y is Hausdorff, then f is closed.*

Proof. Combine the following three topology facts, whose proof we leave as an exercise:
• A closed subset A of a compact space X is compact.
• The continuous image $B := f(A)$ of a compact set A is again compact.
• A compact subset B of a Hausdorff space Y is closed. $\square$

Proof of the Proposition. We need to show that f is a homeomorphism; in fact, we need to show f^{-1} is continuous.

We use two facts:
• By definition, closed sets are complements of open sets. For a subspace $X \subset N$ with the subspace topology, a subset $A \subset X$ is closed if there is a closed set $B \subset N$, such that $A = B \cap X$.
• A mapping $f\colon M \to N$ is continuous if and only if closed sets in N have preimages which are closed in M.

By injectivity, $f^{-1}\colon f(M) \to M$ exists. By the lemma, if $A \subset M$ is closed then $f(A)$ is closed in N. By our first fact this means $f(A)$ is closed in $f(M)$ as well. Thus f^{-1} has the property that the preimages of closed sets are closed. By the second fact, f^{-1} is continuous. $\square$

Remark. The Jordan curve theorem states that the homeomorphic image of a circle in $\mathbb{R}^2$ divides $\mathbb{R}^2$ into two components, one of which is compact. By the proposition, it is sufficient to assume the circle is mapped continuously and injectively into the plane.

2.6. **Submanifolds.** There are various ways to define n-dimensional submanifolds of Euclidean space $\mathbb{R}^{n+k}$ locally:
(i) Inverse image of a regular value of an $\mathbb{R}^k$-valued function.
(ii) Image of a *slice* $U^n \times \{0\} \subset \mathbb{R}^{n+k}$ where $U \subset \mathbb{R}^n$ is open and $U^n \times (-\varepsilon,\varepsilon)^k$ parameterizes an open set in ambient space diffeomorphically,
(iii) parameterization (immersion) with $U \subset \mathbb{R}^n$.
These characterizations all generalize to define submanifolds of ambient manifolds. Here, we choose to turn the second characterization into a definition:

Definition. A subset $M \subset N^{n+k}$, $k \geq 0$, is an n-dimensional *submanifold [Untermannigfaltigkeit]* of N if at each $p \in M$ there is a chart $y\colon V \to \mathbb{R}^{n+k}$ of N subject to

$$y(M \cap V) = y(V) \cap \big(\mathbb{R}^n \times \{0\}\big). \tag{13}$$

That is, the chart y maps the submanifold M locally to a slice. The fact that all points of M within the set V are mapped to the slice can be regarded to say that a submanifold keeps a distance to itself.

A submanifold is a manifold in its own right. Restricting charts (y_α, V_α) satisfying (13) to their first n components we obtain charts $(x_\alpha,\ U_\alpha := V_\alpha \cap M)$ for M with

$$x_\alpha\colon V_\alpha \cap M \to \mathbb{R}^n, \qquad x_\alpha(p) = \big(y_\alpha^1(p), \dots, y_\alpha^n(p)\big).$$

Indeed, these charts certainly cover M, and the transition maps $x_\beta \circ x_\alpha^{-1} = y_\beta \circ y_\alpha^{-1}$ are differentiable as restrictions. Moreover, the set M inherits the Hausdorff and second countability property from N.

We have the following result:

Theorem 10. *If $f\colon M \to N$ is an embedding then its image $f(M) \subset N$ (with the subspace topology) is a submanifold of N.*

Proof. We need to show that each point of $f(M)$ has a neighbourhood V such that a chart (y, V) maps $f(M) \cap V$ to the slice $y(V) \cap (\mathbb{R}^n \times \{0\})$.

As a result of Corollary 9, in terms of the charts (x, U) at p and $(y := \tilde{y}, \tilde{V})$ at $f(p)$ we have

$$y(f(U)) = y(\tilde{V}) \cap \big(\mathbb{R}^n \times \{0\}\big).$$

The set $\tilde{V}$, however, possibly contains points of $f(M \setminus U)$, so that the left hand side could be a proper subset of $y(f(M) \cap \tilde{V})$ (instead of the equality we need).

By assumption, f is a homeomorphism of M onto its image $f(M)$. Thus the open set $U \subset M$ has an image $f(U)$ in N which is open with respect to the subspace topology. That is, there exists an open set $W \subset N$ such that $f(U) = f(M) \cap W$. If we set $V := W \cap \tilde{V}$ then $f(M) \cap V = f(U)$. Thus $y(f(M) \cap V) = y(f(U))$, and this equals $y(V) \cap \big(\mathbb{R}^n \times \{0\}\big)$ since $f(U)$ is contained in V. □

3. The Whitney embedding theorem

In 1944 Whitney proved: Any differentiable n-manifold M can be embedded into $\mathbb{R}^{2n}$. Hence the class of abstract manifolds is no larger than the class of submanifolds of Euclidean space! This result seems to indicate that there is no gain in introducing abstract manifolds. However, for many manifolds it is as hard as superficial to come up with an explicit embedding. This certainly applies to quotient constructions: Even for simple examples such as the Klein bottle or $\mathbb{R}P^2$, embeddings into $\mathbb{R}^4$ are not obvious (see problems, however).

In fact, we will provide the proof only for a slightly weaker result, namely we assume:
• The target is $\mathbb{R}^{2n+1}$ and not $\mathbb{R}^{2n}$. In this form Whitney proved the result first, in 1936.
• The manifold M is compact and so has a finite atlas.

The compactness assumption saves a little work, but the ideas we will explain can also deal with the non-compact case (see [Lee], Chapter 10). However, the relaxed target dimension makes the proof substantially easier. To see why, let us state two main steps of the proof:
(I) We can change an arbitrary differentiable map $f\colon M^n \to \mathbb{R}^{2n}$ to an immersion (choose f to be constant to begin with).
(E) We can change an immersion into $M^n \to \mathbb{R}^{2n+1}$ to an embedding.
Both changes can be achieved by small deformations, i.e., with a small change of L^∞-norm.

The example of M a curve ($n = 1$) with a double point in $\mathbb{R}^3$ can illustrate why part (E) is much more obvious for $\mathbb{R}^{2n+1}$ than for $\mathbb{R}^{2n}$: For a curve in $\mathbb{R}^3$ with a double point we can use the extra dimension to move locally one branch of the curve off the other branch, so that the curve becomes embedded; this change can be made arbitrarily small (in distance, say). However, in $\mathbb{R}^2$, we would need to change the curve globally. This problem is harder and Whitney invented the so-called Whitney trick to achieve the global change. See the Wikipedia entry *Whitney embedding theorem* for an illustration of the need for a global change in the case of curves.

3.1. **Sets of measure zero.** To perturb a given map in step (I) we will show that the immersed maps are dense in the space of differentiable maps, by showing that the "non-immersions" form a set of measure zero.

We consider the Lebesgue measure λ on Euclidean space. A set $A \subset \mathbb{R}^n$ has *measure zero [Nullmenge]* if for each $\varepsilon > 0$ there are countable many measurable sets $\{B_i\}$ which cover, $A \subset \bigcup_{i\in\mathbb{N}} S_i$, and have total measure $\sum_{i\in\mathbb{N}} \lambda(B_i) < \varepsilon$. Specifically, we can and will assume that the sets B_i are balls. For instance, the coordinate subspaces $\mathbb{R}^n \times \{0_k\} \subset \mathbb{R}^{n+k}$ have measure zero for $k \geq 1$.

Proposition 11. (*i*) *Suppose $U, V \subset \mathbb{R}^n$ are open and $f \in C^1(U, V)$. If $A \subset U$ has measure zero then $f(A)$ has measure zero.*
(*ii*) *Suppose $k > 0$ and $U \subset \mathbb{R}^n$, $V \subset \mathbb{R}^{n+k}$ are open. If $f \in C^1(U, V)$ then $f(U)$ has measure zero.*
(*iii*) *Suppose M is an n-dimensional manifold, $k > 0$, and $f\colon M \to \mathbb{R}^{n+k}$ is differentiable. Then $f(M)$ has measure 0. In particular its complement $\mathbb{R}^{n+k} \setminus f(M)$ is dense.*

Example. Consider the countably many intervals in the plane $A := (\mathbb{Q}\cap[0,1])\times[0,1] \subset \mathbb{R}^2$. Using an enumeration of the rationals in $[0,1]$, we can construct a differentiable curve $c\colon \mathbb{R} \to \mathbb{R}^2$ whose image contains A. Then c is an example for (*ii*) or (*iii*): The set $c(\mathbb{R})$ is a set of measure zero and has a dense complement.

Note that the proposition fails to hold for continuous mappings. Indeed, a space filling curve $c(t)$ can map the unit interval $[0,1]$ onto the square $[0,1]\times[0,1]$, thereby giving a counterexample to (ii) and also (i).

Proof. (i) Let the open balls $\{B_i : i \in \mathbb{N}\}$ cover A with total measure less than $\varepsilon > 0$. As Case 1 let us consider the case that all B_i are contained in a compact subset $K \subset U$. Over K the map f has a uniformly bounded differential $\|df_x\| \le C = C(K)$, in particular $f|_K$ is Lipschitz. Thus for balls we have $f(B_r(p)) \subset B_{rC}(f(p))$. We conclude

$$\lambda(f(A)) \le \lambda\Big(f\big(\bigcup_i B_i\big)\Big) = \lambda\Big(\bigcup_i f(B_i)\Big) \le \sum_i \lambda\big(f(B_i)\big) \le C^n \sum_i \lambda(B_i) < C^n\varepsilon. \tag{14}$$

As Case 2 we consider the general case. The open set U has an *exhaustion [Ausschöpfung]* with a family $(K_j)_{j\in\mathbb{N}}$ of compact sets

$$K_1 \subset \cdots \subset K_j \subset \cdots \subset \bigcup_{\ell\in\mathbb{N}} K_\ell = U.$$

For instance, the following exhausting sets are bounded and closed:

$$K_j := \big\{x \in U : |x| \le j \text{ and } B_{1/j}(x) \subset U\big\}.$$

Now consider the subsets $A_j := A \cap K_j$ of A. Each A_j has a countable cover with the measurable sets $T_{ij} := B_i \cap K_j \subset K_j$, and as in Case 1 we have $\lambda(f(T_{ij})) \le (C_j)^n\lambda(B_i)$ where C_j is a bound on the differential $\|df\|$ over the compact set K_j. Thus we can adapt the estimate (14) of Case 1 to say

$$\lambda(f(A_j)) \le \lambda\Big(\bigcup_i f(T_{ij})\Big) \le (C_j)^n \sum_i \lambda(B_i) < (C_j)^n\varepsilon \qquad \text{for all } j.$$

That is, A_j is a set of measure zero for each j, and therefore the countable union $A = \bigcup A_j$ is also a set of measure zero.

(ii) Extend f to a differentiable map $F\colon U\times\mathbb{R}^k \to \mathbb{R}^{n+k}$ by setting $F(x,y) := f(x)$. Then $A := U\times\{0_k\} \subset \mathbb{R}^n\times\{0_k\}$ is a set of measure 0 in $\mathbb{R}^{n+k}$, and so by (i) its image $F(A) = f(U)$ has measure 0 in $\mathbb{R}^{n+k}$.

(iii) Consider charts $x_\alpha\colon U_\alpha \to \mathbb{R}^n$ of a countable atlas of the manifold M, where $\Omega_\alpha := x_\alpha(U_\alpha) \subset \mathbb{R}^n$ is open. Then from (ii) we have $\lambda\big(f(U_\alpha)\big) = \lambda\big((f\circ x_\alpha)(\Omega_\alpha)\big) = 0$, and thus $\lambda\big(f(M)\big) = \lambda\big(\bigcup_\alpha f(U_\alpha)\big) \le \sum_\alpha \lambda\big(f(U_\alpha)\big) = 0$, independently of the atlas chosen. ☐

Remark. By (i) the measure zero property of sets is invariant under diffeomorphism. This can be used to define sets of measure zero on manifolds, independently of the choice of atlas.

3.2. **Partitions of unity.** An *(open) covering [Überdeckung]* of a topological space M is a family of open sets $\{U_\alpha : \alpha \in A\}$ such that $\bigcup_{\alpha\in A} U_\alpha = M$.

Definition. A *partition of unity [Zerlegung der Eins]* subordinate to an open covering $\{U_\alpha : \alpha \in A\}$ of a manifold M is a family of differentiable functions $\varphi_\alpha \colon M \to [0,1]$, $\alpha \in A$, such that
(i) $\operatorname{supp}\varphi_\alpha = \overline{\{x \in M : \varphi_\alpha(x) \neq 0\}} \subset U_\alpha \ \forall\alpha$,
(ii) $\forall p \in M$ there is a neighbourhood $W(p)$ which meets only finitely many $\operatorname{supp}\varphi_\alpha$,
(iii) $\sum_{\alpha\in A} \varphi_\alpha(p) = 1 \ \forall p \in M$.

Note that (ii) implies that for each p the sum in (iii) is finite. Condition (ii) clearly holds when the covering is locally finite, that is, each $p \in M$ is contained in at most finitely many U_α.

From second countability, we have the following consequence which we state without proof:

Theorem 12. *A differentiable manifold M has a partition of unity subordinate to any open covering $\{U_\alpha : \alpha \in A\}$.*

In fact, for a topological space the same existence statement is equivalent to the space being Hausdorff with a countable basis.

For $U \subset M$, let us call a differentiable function $\psi \colon M \to [0,1]$ with $\operatorname{supp}\psi \subset U$, a *bump function [Hutfunktion] relative to U*. We will use a partition of unity in form of the bump functions of the following statement (ii):

Corollary 13. (i) *Given a closed set A and an open set U with $A \subset U \subset M$ there is a bump function $\psi \colon M \to [0,1]$ relative to U with $A \subset \psi^{-1}(1)$.*
(ii) *Given a covering $\{U_\alpha : \alpha \in A\}$ of M, there exists a family of bump functions ψ_α relative to U_α such that the closed sets $C_\alpha := \{x \in U_\alpha : \psi_\alpha(x) = 1\}$ still cover M.*

Proof. (i) The sets $U_1 := U$ and $U_2 := M \setminus A$ form an open covering of M. Let $\{\varphi_1, \varphi_2\}$ be a partition of unity subordinate to it. Then $\varphi_2 = 0$ on A and so $\psi := \varphi_1$ must be 1 on A.

(ii) Let $\{\varphi_\alpha\}$ be a partition of unity subordinate to the sets $\{U_\alpha\}$. Then the closed support sets $A_\alpha := \operatorname{supp}\varphi_\alpha \subset U_\alpha$ still cover M. Apply part (i) to $A_\alpha \subset U_\alpha \subset M$ in order to obtain bump functions ψ_α with support in U_α. Then $A_\alpha \subset \psi_\alpha^{-1}(1) = C_\alpha$, and so the C_α also cover M. $\square$

3.3. **Embedding part.** It appeals to intuition that an immersion of an n-dimensional manifold in $\mathbb{R}^{2n+1}$ can be perturbed to an embedding: At a self-intersection of two "leaves" of the manifold, a small perturbation suffices to move one leaf off the other.

Proposition 14. *Let M be compact and $f\colon M \to \mathbb{R}^{2n+1}$ be an immersion. Then for each $\varepsilon > 0$ there is an embedding $h\colon M \to \mathbb{R}^{2n+1}$ such that $\|h - f\|_\infty = \sup_M |h - f| < \varepsilon$.*

Proof. By Thm. 8 locally an immersion is an embedding. Thus M has a covering with charts $\{(x_\alpha, U_\alpha)\}$ such that f restricted to U is an embedding. Since M is compact, a finite number $U_1, \ldots, U_\ell$ of such charts suffices to cover. Corollary 13(ii) then yields bump functions $\psi_1, \ldots, \psi_\ell$ such that the closed sets $C_k := \psi_k^{-1}(1) \subset U_k$ cover. The sets C_i are compact as a closed sets contained in a compact space, as asserted on p.16.

It will be convenient to set $h_0 := f$ and $C_0 := \emptyset$, and to assume $\varepsilon < 1$. For $k = 1, \ldots, \ell$ we will verify recursively there is a choice of vectors $b_k \in \mathbb{R}^{2n+1}$ with $|b_k| < 1$, such that the perturbed functions

$$h_k\colon M \to \mathbb{R}^{2n+1}, \qquad h_k(p) := h_{k-1}(p) + \frac{\varepsilon}{\ell}\psi_k(p)\, b_k \tag{15}$$

are subject to the following conditions:
1. h_k is an immersion of M,
2. h_k restricted to U_i is injective for each $i = 1, \ldots, \ell$,
3. h_k is injective on $C_0 \cup C_1 \cup \cdots \cup C_k$.

Note that 1. and 2. preserve the conditions which $f = h_0$ satisfies, while 3. is meant to extend injectivity recursively from C_1 to all of M.

For $k := \ell$, conditions 1. and 3. prove the theorem. Indeed, $h := h_\ell$ is an injective immersion of the entire manifold M, which by the proposition of Sect. 2.5 is an embedding. Moreover, we have, as desired, that indeed h is a small perturbation of f,

$$\|h - f\|_\infty \le \|h_\ell - h_{\ell-1}\|_\infty + \cdots + \|h_1 - h_0\|_\infty \le \sum_{k=1}^{\ell} \frac{\varepsilon}{\ell}\|\psi_k\|_\infty\, |b_k| < \varepsilon.$$

Now we start with the recursive proof. Note that already for $k = 0$ conditions 1. to 3. hold as f is an immersion, f restricted to all U_i is injective, and f restricted to C_0 is vacuous. So it suffices to show that for $k \ge 1$ the assumption that h_{k-1} satisfies conditions 1. to 3. allows us to choose b_k such that 1. to 3. hold for h_k as in (15).

1. What follows could be summarized by saying that immersions of compact sets are preserved under small perturbation. Since h_{k-1} is an immersion of C_i, for each $p \in C_i$ there is an $n \times n$-submatrix of the Jacobian $d(h_{k-1} \circ x_i^{-1})$ with nonzero determinant, that is,

$$F^i_{k-1}(p) := \max_{n\times n \text{ submatrices}} \left|\mathrm{minor}(d(h_{k-1} \circ x_i^{-1})(x_i(p)))\right| > 0 \qquad \text{for all } p \in C_i.$$

Since C_i is compact and F^i_{k-1} is continuous there is $c = c(i) > 0$ with $F^i_{k-1}(p) \ge c$ for all $p \in C_i$. Replacing $c(i)$ by the minimum $c > 0$ of the finitely many $c(i)$ we arrive at $F^i_{k-1}(p) \ge c$ for all $p \in M$.

Let us consider F^i_k as a function of both $p \in C_i$ and b_k. This function is continuous, and for $b_k = 0$ bounded below by c. Since C_i is compact this implies

$$\text{there exists } \delta_k = \delta_k(i) > 0 \text{ such that if } |b_k| \le \delta_k \text{ then } F^i_k \ge \frac{c}{2},$$

and so h_k restricted to C_i is an immersion. To obtain the same result for the entire manifold M, simply choose $\delta_k := \min_i \delta_k(i)$.

Before proving 2. and 3. let us verify the following claim: For each k, we can find a *good* b_k with $|b_k| < \delta_k$, defined by the implication

$$\text{(X)} \qquad \text{if } \psi_k(p) \neq \psi_k(q) \text{ then also } h_k(p) \neq h_k(q) \quad \text{for all } p, q \in M.$$

We solve $h_{k-1}(p) + \frac{\varepsilon}{\ell}\psi_k(p)\, b_k \neq h_{k-1}(q) + \frac{\varepsilon}{\ell}\psi_k(q)\, b_k$ for b_k to see this that b_k is good is equivalent to $b_k \notin F(V)$ where the mapping F is given by

$$F\colon\ V := \{(p,q) \in M \times M : \psi_k(p) \neq \psi_k(q)\} \to \mathbb{R}^{2n+1}, \qquad F(p,q) := -\frac{\ell}{\varepsilon}\,\frac{h_{k-1}(p) - h_{k-1}(q)}{\psi_k(p) - \psi_k(q)}.$$

Now the product $M \times M$ is a $2n$-dimensional manifold, and so is its open subset V. Thus the image $F(V)$ under the differentiable map F has measure zero in $\mathbb{R}^{2n+1}$ and so a dense complement, by Proposition 11(*iii*). Therefore, subject to $0 < |b_k| < \delta_k$, we can indeed find a good $b_k \notin F(V)$.

To prove 2. and 3. recursively, it will be useful to state a case somewhat complementary to (X), but with an additional condition:

$$\text{(Y)} \qquad \text{If } \psi_k(p) = \psi_k(q) \text{ and } h_{k-1}(p) \neq h_{k-1}(q) \text{ for } p, q \in M \text{ then } h_k(p) \neq h_k(q).$$

2. For $p, q \in U_i$ the recursive assumption 2. shows the assumption $h_{k-1}(p) \neq h_{k-1}(q)$ of (Y) is fulfilled. By definition of h_k (see (15)) this proves that (Y) holds. Then (X) and (Y) together prove 2. for h_k.

3. To make (X) and (Y) a complementary case distinction, we need to verify the extra assumption in (Y),

$$\text{(16)} \qquad h_{k-1}(p) \neq h_{k-1}(q) \qquad \text{for } p \neq q \in C_0 \cup C_1 \cup \cdots \cup C_k,$$

will hold, given that $\psi_k(p) = \psi_k(q)$. To prove (16) let us again distinguish cases:
• Suppose p, q are both in C_k. In view of $C_k \subset U_k$ then (16) is implied by Hypothesis 2.
• If $p \in C_k$ but $q \notin C_k$ (or vice versa) then $\psi_k(p) = 1$ but $\psi_k(q) < 1$, so that the other assumption of (Y) is not satisfied.
• Otherwise $p, q \in C_0 \cup \cdots \cup C_{k-1}$. Then the recursive hypothesis 3. implies (16). ☐

To digest the preceding proof, let me ask how the dimension assumption enters, and in which sense the proof contains the idea of moving one leaf off another at intersections.

Remark. To make the proof work for non-compact M, an exhaustion with compact sets $(C_i)_{i\in\mathbb{N}}$ can be used. The essential change then is to replace ℓ by 2^i in (15) so that $\|h-f\|_\infty$ becomes a converging infinite sum.

3.4. **Matrices of fixed rank.** A version of the perturbation idea used for the embeddedness part lets us also perturb a given differentiable map from M into $\mathbb{R}^{2n}$ –perhaps a constant– into an immersion. To do this in the next subsection we will perturb the given map on each chart by adding a map which is linear in the coordinates and piece the result together using a partition of unity.

In order to obtain an immersion we have to make sure that the Jacobian of our perturbation avoids the singular matrices whose rank is less than n. For codimension at least n this will follow from a computation of the codimension of the space of singular matrices in the space of all matrices. This is the content of the present subsection.

Let $\mathbb{R}^{m\times n}$ denote the space of the real m by n matrices. We identify these matrices with Euclidean space, $\mathbb{R}^{m\times n}=\mathbb{R}^{mn}$.

Proposition 15. *For each $0\le r\le\min\{m,n\}$, the space of rank r matrices*

$$\mathcal{M}_r := \left\{M\in\mathbb{R}^{m\times n} : \operatorname{rank} M = r\right\}$$

is a submanifold of $\mathbb{R}^{mn}$ with

$$\dim\mathcal{M}_r = r(n+m-r). \tag{17}$$

As expected, the dimension formula is symmetric in m,n, and for the case of maximal rank $r=\min\{m,n\}$, the dimension is mn.

Proof. Representing matrices as block matrices, we introduce the set

$$U := \left\{M=\begin{pmatrix}A & B\\ C & D\end{pmatrix}\in\mathbb{R}^{m\times n} : A\in\mathbb{R}^{r\times r} \text{ satisfies } \det A\neq 0\right\}.$$

U is an open subset of $\mathbb{R}^{m\times n}$ since the determinant is continuous.

Matrices $M\in U$ have rank at least r. Let us now derive a condition for M to have rank exactly r. We transform M into a standard echelon form *[Stufenform]* by multiplying it from the right with a suitable invertible $n\times n$-matrix in block matrix form:

$$\begin{pmatrix}A & B\\ C & D\end{pmatrix}\begin{pmatrix}A^{-1} & -A^{-1}B\\ 0 & 1_{n-r}\end{pmatrix}=\begin{pmatrix}1_r & 0\\ CA^{-1} & D-CA^{-1}B\end{pmatrix}\in U$$

Clearly,

$$\operatorname{rank}\begin{pmatrix}A & B\\ C & D\end{pmatrix}=r \quad\Leftrightarrow\quad D-CA^{-1}B=0.$$

This gives rise to an implicit description of $\tilde{U} := U \cap \mathcal{M}_r$ as a set $\Phi^{-1}(0)$ where, specifically, Φ is the smooth function

$$\Phi\colon U \to \mathbb{R}^{(m-r)\times(n-r)}, \qquad \Phi(\begin{pmatrix} A & B \\ C & D \end{pmatrix}) := D - CA^{-1}B.$$

To show that $\tilde{U}$ is a submanifold of $\mathbb{R}^{nm}$ we claim that the 0-matrix is a regular value of Φ. We need to show that the differential $d\Phi$ is surjective, when taken at a matrix $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Phi^{-1}(0)$. Let us prove that any $S \in \mathbb{R}^{(m-r)\times(n-r)}$ is attained: The curve

$$M(t) := \begin{pmatrix} A & B \\ C & D+tS \end{pmatrix} \in U \quad \text{satisfies} \quad \Phi\big(M(t)\big) = (D - CA^{-1}B) + tS,$$

and so the linearization $\frac{d}{dt}\Phi\big(M(t)\big)|_{t=0}$ is

$$d\Phi_{\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)}(\begin{pmatrix} 0 & 0 \\ 0 & S \end{pmatrix}) = S.$$

So indeed $d\Phi$ is surjective at each point of $\Phi^{-1}(0)$, as desired.

Now pick an arbitrary $M \in \mathcal{M}_r$. Then some $(r\times r)$-submatrix of M has rank r. Analoguous to the definition of U, we let V be the open set of all matrices $\mathbb{R}^{m\times n}$ such that this particular submatrix has rank r. Moreover, we let $\tilde{V} := V \cap \mathcal{M}_r$. Let us reindex coordinates in $\mathbb{R}^n$ and $\mathbb{R}^m$ such that the submatrix under consideration is mapped to the top left $r \times r$ submatrix. This gives a map $\Psi\colon \tilde{V} \subset \mathbb{R}^{mn} \to \tilde{U} \subset \mathbb{R}^{mn}$ which is a diffeomorphism onto its image. Thus $\tilde{V} = (\Phi \circ \Psi)^{-1}(0)$ is a submanifold of $\mathbb{R}^{m\times n} = \mathbb{R}^{mn}$.

Since this applies to each choice of submatrix, we have covered $\mathcal{M}_r$ with finitely many submanifolds, each of them having an implicit description. That is, $\mathcal{M}_r$ is a submanifold altogether. Moreover, subtracting the number of constraints from the full space dimension we verify

$$\dim \mathcal{M}_r = \dim \mathbb{R}^{m\times n} - \dim \mathbb{R}^{(m-r)\times(n-r)} = mn - (m-r)(n-r) = (m+n-r)r. \qquad \square$$

It can be shown that M_r is connected except for the case $r = m = n$ when there are two components corresponding to the sign of the determinant.

For the immersion problem, the singular matrices $\mathcal{S}$ of rank at most $n-1$ are to be avoided by the Jacobian of the desired map. We will guarantee this property by choosing m large enough, namely $m \geq 2n$:

Corollary 16. *The dimension of the submanifolds $\mathcal{M}_r \subset \mathbb{R}^{m\times n}$ satisfies*

$$\dim \mathcal{M}_0 \leq \cdots \leq \dim \mathcal{M}_{n-1} = mn - m + n - 1.$$

That is, the singular matrices have codimension $m-n+1$ in the mn-dimensional space of all matrices. We will need that this number is at least $n-1$. The smallest value of m for which this holds is $m=2n$, which will be our target dimension.

Proof. To prove $0=\dim\mathcal{M}_0 \le \cdots \le \dim\mathcal{M}_n$ we verify for r real:

$$r \le n \le m \quad \Rightarrow \quad \frac{d}{dr} r(n+m-r) = n+m-2r \ge 0.$$

Moreover, (17) gives for $r=n-1$ the desired formula, as

$$\dim\mathcal{M}_{n-1} = (n-1)\big(m+n-(n-1)\big) = (n-1)(m+1).$$

□

3.5. **Immersion theorem.** We want to perturb a given differentiable map f to an immersion h. To exhibit the essential idea of the proof we formulate the main point first:

Lemma 17. *Suppose (x,U) is a chart of M^n and $f\colon M^n \to \mathbb{R}^{2n}$ is differentiable. Then for any $\delta>0$, there exists a matrix $A\in\mathbb{R}^{2n\times n}$, $|A|\le\delta$ such that*

$$h\colon M^n \to \mathbb{R}^{2n}, \qquad h(p) := f(p) + Ax(p), \tag{18}$$

is an immersion.

Note that h changes f by an amount linear in the coordinates.

Proof. To write the map in terms of $x(U)$ we consider $\tilde h\colon x(U)\to\mathbb{R}^{2n}$, where

$$\tilde h(u) := (h\circ x^{-1})(u) = (f\circ x^{-1})(u) + Ax(x^{-1}(u)) =: \tilde f(u) + Au.$$

Since charts are diffeomorphisms, it is equivalent that h or $\tilde h$ is an immersion. That is, we need to prove:

$$\exists A: \quad d\tilde h_u = d\tilde f_u + A \qquad \text{has rank } n \text{ for each point } u\in x(U), \tag{19}$$

or equivalently, $d\tilde h_u \notin \mathcal{M}_0\cup\cdots\cup\mathcal{M}_{n-1}$ for all $u\in x(U)$.

To establish the existence of A, consider the map

$$F\colon \mathbb{R}^{2n\times n}\times x(U) \to \mathbb{R}^{2n\times n}, \qquad F(B,u) := B - d\tilde f_u.$$

We claim we can choose A with $|A|<\delta$ disjoint from the image of all singular matrices B (and so making it impossible for $B=d\tilde h_u$ to have rank less than n),

$$A \ \notin\ F\big(\mathcal{M}_0\times x(U)\big) \cup \cdots \cup F\big(\mathcal{M}_{n-1}\times x(U)\big). \tag{20}$$

For each $r\le n-1$ the manifold $\mathcal{M}_r$ has dimension at most $2n^2-n-1$ by Corollary 16. Thus the product manifold $\mathcal{M}_r\times x(U)$ has dimension at most $(2n^2-n-1)+n = 2n^2-1$. Since the target space $\mathbb{R}^{2n\times n}$ is $2n^2$-dimensional, Proposition 11(iii) is applicable with codimension $k \geq 1$. Consequently each image $F(\mathcal{M}_r \times x(U))$ is a set of measure zero in $\mathbb{R}^{2n\times n}$. The dense complement of the union of these sets contains an $A \in \mathbb{R}^{2n\times n}$ with norm less than δ, proving the claim. For such A then (20) and (19) hold, that is, h is an immersion. □

We formulate the immersion statement for the entire manifold:

Proposition 18. *Suppose M is compact, $f\colon M \to \mathbb{R}^{2n}$ is differentiable, and $\varepsilon > 0$. Then there exists an immersion $h\colon M \to \mathbb{R}^{2n}$ such that $\|f-h\|_\infty = \sup_M |f-h| < \varepsilon$.*

Proof. Choose a finite atlas $(x_1, U_1), \dots, (x_\ell, U_\ell)$. We assume the charts take image in the unit ball $B_1 \subset \mathbb{R}^n$ by composing with a diffeomorphism $\mathbb{R}^n \leftrightarrow B_1$ if necessary. Corollary 13(*ii*) gives bump functions ψ_k relative to the U_k, such that the closed sets $C_k = \operatorname{supp}\psi_k \subset U_k$ still cover M; we also set $C_0 := \emptyset$.

We construct functions $h_1, \dots, h_\ell =: h$, by perturbing f in each chart linearly as in the lemma. Using the bump functions we can piece the result together. Setting, morever, $h_0 := f$, we therefore set for $k = 1, \dots, \ell$,

$$h_k(p) := h_{k-1}(p) + \frac{\varepsilon}{\ell}\psi_k(p) A_k x_k(p)$$

where $A_k \in \mathbb{R}^{2n\times n}$ with $|A_k| \leq 1$ is yet to be determined. Here we assume that x_i has been extended with value 0 to all of M; the products $\psi_i x_i$ are then still differentiable.

We claim the following for $k = 1, \dots, \ell$:

$$\text{(21)}\quad \exists A_k \quad \text{such that} \quad h_k = h_{k-1} + \frac{\varepsilon}{\ell}\psi_k A_k x_k \quad \text{is an immersion on } C_0 \cup C_1 \cup \cdots \cup C_k.$$

For $k = \ell$ this says that $h := h_\ell$ is an immersion on all of M, with $\|f-h\|_\infty < \varepsilon$ as desired.

Note that (21) holds for $k = 0$, where it is vacuous. For $k \geq 1$ let us now prove recursively the step $k-1 \mapsto k$.

• For points $p \in C_0 \cup \ldots \cup C_{k-1}$: We can reason as in the proof of Proposition 14 (under 1.): Again $dh_k(p)$ depends continuously on $dh_{k-1}(p)$ and A_k, and $C_0 \cup \cdots \cup C_{k-1}$ is compact. Thus there exists $\delta_k \in (0,1)$ such that for all matrices A_k with $|A_k| < \delta_k$ the map $dh_k(p)$ has rank n.

• For $p \in C_k$: Lemma 17 shows we can find A_k that we can achieve $h_{k-1}(p) + \frac{\varepsilon}{\ell} A_k x_k(p)$ to have rank n for all $p \in C_k$. where the matrix A_k is subject to our constraint $|A_k| \leq \delta_k$. Since $\psi_k = 1$ on C_k this proves $h_k(p)$ is an immersion. □

Remark. Using more subtle arguments Whitney showed in 1944 that for $n \geq 2$ each n-manifold can actually be immersed into $\mathbb{R}^{2n-1}$

If we choose $f \equiv 0$ in Proposition 18, and insert the resulting immersion into Proposition 14 we obtain:

Theorem 19. *Every compact n-manifold admits an embedding to $\mathbb{R}^{2n+1}$.*

All our arguments generalize directly to arbitrary, not necessarily compact, manifolds. For that case, countably many charts have to be considered, and the recursive definitions of h_k must be chosen with a decreasing factor in order to guarantee convergence of the infinite sums. See, for instance, Lee [L].

Let us state a consequence of the embedding theorem. The distance function $d(x, y) = |x - y|$, pulled back to M via the embedding $M \hookrightarrow \mathbb{R}^{2n+1}$, provides a metric on M. We conclude:

Corollary 20. *Each compact differentiable manifold M carries a metric d (consistent with its topology) such that (M, d) is a metric space.*

Part 2. Vector fields, flows and the Frobenius theorem

4. Vector fields

Vector fields are essential objects in order to study differential manifolds. In this section, we will see vector fields in three different roles: As geometric vector fields, as directional (or Lie) derivatives, and defining an ordinary differential equation.

4.1. Geometric vector fields.

Definition. A *(differentiable) vector field* X on a manifold M is a differentiable mapping

$$X\colon M \to TM \qquad \text{such that} \qquad X(p) \in T_pM.$$

We let $\mathcal{V}(M)$ denote the vector space of all vector fields on M.

Let (x, U) be a chart. Then the vector field X has a *principal part* $\xi\colon U \to \mathbb{R}^n$ w.r.t. (x, U). The standard basis $e_i(p) = [x^{-1}\big(x(p) + tb_i\big)]$ w.r.t. (x, U) can be used to represent $X|_U$ in terms of ξ as in (7):

$$X(p) = \sum_{i=1}^n \xi^i(p)e_i(p) \qquad \text{for all } p \in U.$$

By definition, $X\colon M \to TM$ is differentiable if and only if all its chart representations are differentiable, namely the maps

$$y \circ X \circ x^{-1}\colon x(U) \to y(TM), \qquad u \mapsto y\big(X(x^{-1}(u))\big) = \Big(x\big(x^{-1}(u)\big) = u,\ \xi\big(x^{-1}(u)\big)\Big).$$

Since the identity and x^{-1} are differentiable anyway, this is equivalent to all principal parts ξ being differentiable.

Examples. 1. The tori T^n have a basis of non-vanishing vector fields (they are *parallelizable*). On the other hand, for surfaces the Poincaré-Hopf theorem says that neither $\mathbb{S}^2$ nor a surface of genus $g \geq 2$ (a surface with g "holes") carries a vector field without a zero.

2. For $\mathbb{R}^n$ we identified $T_p\mathbb{R}^n$ with $\mathbb{R}^n$ in (9), namely equivalence classes of curves with principal parts (8). Similarly we identify $X \in \mathcal{V}(\mathbb{R}^n)$ with its principal part $\xi\colon \mathbb{R}^n \to \mathbb{R}^n$.

4.2. **Lie derivative.** We first consider the case of a single tangent vector. Recall that in $\mathbb{R}^n$ a curve c through p with tangent vector $\xi := c'(0)$ induces a directional derivative *[Richtungsableitung]* of $f\colon \mathbb{R}^n \to \mathbb{R}$, namely

$$\partial_\xi f(p) := \frac{d(f \circ c)}{dt}(0) = \sum_{i=1}^n \frac{\partial f}{\partial x^i}\big(c(0)\big)\frac{d}{dt}c^i(0) = \sum_{i=1}^n \frac{\partial f}{\partial x^i}(p)\,\xi^i. \tag{22}$$

For a manifold M, let us denote the set of all real-valued differentiable functions with $C^\infty(M) := C^\infty(M, \mathbb{R})$. For $f \in C^\infty(M)$ the function $f \circ c$ still is real-valued on an interval, and so we can define a directional derivative as in the Euclidean case:

Definition. The *Lie derivative* of $f \in C^\infty(M)$ at $p \in M$ in direction $v = [c] \in T_pM$ is

$$\partial_v f := \frac{d(f \circ c)}{dt}(0).$$

Other common notation for the Lie derivative includes $v(f)$, $L_v f$.

To show the Lie derivative is well-defined, that is, independent of the representative c of v, we calculate w.r.t. a chart (x, U) at p:

$$\frac{d(f \circ c)}{dt}(0) = \frac{d}{dt}\big(f \circ x^{-1} \circ x \circ c\big)(0) = \underbrace{d(f \circ x^{-1})_{x(p)}}_{\text{independent of } c} \underbrace{\frac{d}{dt}\big(x \circ c\big)(0)}_{\text{depends only on } [c]} . \tag{23}$$

Standard calculus rules for the function $f \circ c$ give that the Lie derivative ∂_v is
• $\mathbb{R}$-linear in $C^\infty(M)$, $\partial_v(\lambda f + g) = \lambda\,\partial_v f + \partial_v g$ $\forall \lambda \in \mathbb{R}$, $f, g \in C^\infty(M)$, $v \in T_pM$, and
• satisfies the *product rule* $\partial_v(fg) = f(p)\,\partial_v g + (\partial_v f)g(p)$, for all $v \in T_pM$ and $f, g \in C^\infty(M)$.
An operator with these properties is called a *derivation.* Tangent vectors can be introduced as derivations, with the advantage of avoiding the reference to charts.

Let us give local representations of the Lie derivative. Denote the partial derivative w.r.t. the i-th coordinate in $\mathbb{R}^n$ by ∂_i. Then the right hand side of (23) reads:

$$\partial_v f = \sum_{i=1}^n \partial_i(f \circ x^{-1})\Big|_{(x\circ c)(0)} \frac{d}{dt}(x \circ c)^i(0).$$

Using a notation as introduced for the chain rule at the end of Subsection 2.1, we set

$$\frac{\partial f}{\partial x^i}\Big|_p := \partial_i(f \circ x^{-1})\Big|_{x(p)}. \tag{24}$$

It lets the Lie derivative for manifolds appear as its Euclidean counterpart (22):

$$\partial_v f = \sum_{i=1}^n \frac{\partial f}{\partial x^i}(p)\,\xi^i \qquad \text{where } \xi \text{ is the principal part of } v. \tag{25}$$

Example. The following calculation verifies once again that the Lie derivative agrees for different choices of charts:

$$\sum_{i=1}^n \partial_i(f \circ x^{-1})\big|_{x(p)}\,\xi^i = \sum_{i=1}^n \partial_i\big(f \circ y^{-1} \circ y \circ x^{-1}\big)\Big|_{x(p)}\,\xi^i$$

$$\overset{\text{chain rule}}{=} \sum_j \left(\partial_j(f \circ y^{-1})\big|_{y(p)} \sum_i \partial_i(y \circ x^{-1})^j\big|_{x(p)}\xi^i\right) = \sum_{j=1}^n \partial_j(f \circ y^{-1})\big|_{y(p)}\,\eta^j$$

Employing the notation (24) this reads

$$\sum_{i=1}^{n} \frac{\partial f}{\partial x^i}\xi^i \overset{\text{chain rule}}{=} \sum_{i,j} \frac{\partial f}{\partial y^j}\frac{\partial y^j}{\partial x^i}\xi^i = \sum_j \Big(\frac{\partial f}{\partial y^j}\sum_i \frac{\partial y^j}{\partial x^i}\xi^i\Big) = \sum_{j=1}^{n} \frac{\partial f}{\partial y^j}\eta^j. \tag{26}$$

We now consider the *Lie derivative for vector fields.* If $X \in \mathcal{V}(M)$ and $f \in C^\infty(M)$ then $p \mapsto \partial_{X(p)} f \in \mathbb{R}$ defines a function $\partial_X f \in C^\infty(M)$. Its local expression, in terms of a chart x, is

$$(\partial_X f)(p) = \sum_{i=1}^{n} \xi^i(p)\frac{\partial f}{\partial x^i}(p) \qquad \text{for } f \in C^\infty(M).$$

The Lie derivative ∂_X on a manifold M represents a differential operator of first order, having the following properties:

1. $\partial_X \colon C^\infty(M) \to C^\infty(M)$ is $\mathbb{R}$-linear, satisfying the product rule.
2. $(\partial_X f)(p)$ depends only on $X(p)$, and on f restricted to a neighbourhood of p.
3. We claim an equality for the Lie derivative which is well-known in the Euclidean case:

$$df X = \partial_X f \qquad \text{for all } X \in \mathcal{V}(M),\ f \in C^\infty(M). \tag{27}$$

To verify this identity of functions in $C^\infty(M)$, pick p and represent $X_p = [c]$ with a curve $c\colon I \to M$ where $c(0) = p$. Since Euclidean space and its tangent space are identified, see (9), the "vector" $df_p[c]$ is a real number agreeing with $(f \circ c)'(0)$. We obtain, as desired:

$$df_p[c] \overset{\text{def. differential}}{=} [f \circ c] \overset{(8)}{=} \big(f \circ c\big)'(0) \overset{\text{def. Lie deriv.}}{=} \partial_{[c]} f$$

4. Let $X, Y \in \mathcal{V}(M)$ and suppose $\partial_X f \equiv \partial_Y f$ holds for all $f \in C^\infty(M)$. Then $X = Y$. To see this at p, pick a chart (x, U) and note that locally $\partial_X f = \sum \xi^i \frac{\partial f}{\partial x^i}$ and $\partial_Y f = \sum \eta^i \frac{\partial f}{\partial x^i}$. In view of 2., we may choose the locally defined function $f := x^i \in C^\infty(U)$. Then $\xi^i = \partial_X x^i = \partial_Y x^i = \eta^i$ for each i, and so $X = Y$.

4.3. **Integral curves of a vector field.** As for Euclidean space, a vector field on a manifold poses an ordinary differential equation, which so-called integral curves solve.

A curve $c\colon I \to \mathbb{R}^n$ has the tangent vector $c'(t) \in \mathbb{R}^n = T_{c(t)}\mathbb{R}^n$. For the manifold case, $c\colon I \to M$, we use $c'(t)$ to denote the specific tangent vector

$$c'(t) := dc_t(1) \ \in \ T_{c(t)}M.$$

Note that here dc_t is a linear map, sending tangent vectors to $\mathbb{R}$ (such as 1) to tangent vectors of M. The problem is familiar from 1-dimensional calculus when regarded as a special case of multi-variable calculus: For a map $f\colon \mathbb{R} \to \mathbb{R}$ the linearisation $df_t\colon \mathbb{R} =$

$T_t\mathbb{R} \to \mathbb{R} = T_{f(t)}\mathbb{R}$ maps $s \mapsto df_t(s)$; using matrix notation we write $df_t(s) = (f'(t))s$, where clearly $f'(t)$ is determined by $df_t(1) = f'(t)$.

Given $X \in \mathcal{V}(M)$ and $p \in M$, we want to determine a curve $c\colon I \to M$, where $I \ni 0$ is an open (time) interval, such that c solves the initial value problem

$$c(0) = p, \quad c'(t) = X\big(c(t)\big) \quad \text{for all } t \in I. \tag{28}$$

A solution c is called an *integral curve* of X through p. In case its interval of definition cannot be extended we call c a *maximal solution*. We can regard (28) as an autonomous (non time-dependent) ODE.

Theorem 21. *For each $X \in \mathcal{V}(M)$ and all $p \in M$ there exists a unique maximal solution to the initial value problem* (28)*, defined on an open interval $I \subset \mathbb{R}$.*

Proof. • Local existence: Consider a chart (x, U) at p with $x(U) =: \Omega \subset \mathbb{R}^n$ and let $\xi(p)$ be the principal part of $X(p)$. We must construct a curve c through p defined on some interval $I \ni 0$ such that

$$(x \circ c)'(t) = \xi\big(c(t)\big) = (\xi \circ x^{-1} \circ x \circ c)(t). \tag{29}$$

Let us represent the solution locally: Writing $\tilde{\xi} := \xi \circ x^{-1}$ for a smooth vector field on Ω, and $\gamma := x \circ c$ for the chart representation of the desired curve in Ω we see (29) is equivalent to

$$\gamma'(t) = \tilde{\xi}\big(\gamma(t)\big), \qquad \gamma\colon I \to \Omega, \ \ \gamma(0) = x(p). \tag{30}$$

This is an ordinary differential equation, and so by the Peano existence theorem there is a solution γ to (30), and hence $c = x^{-1} \circ \gamma$ solves (29).

• Independence of choice of charts: Suppose (x, U) and (y, V) are two charts with non-empty intersection $U \cap V$. Consider any $q \in U \cap V$ and two solutions $c_1\colon I_1 \to U$ and $c_2\colon I_2 \to V$, agreeing at the point $c_1(t_0) = c_2(t_0) = q \in U \cap V$. The restrictions of c_1, c_2 to $I_1 \cap I_2$ take values in $U \cap V$ and satisfy the same local ODE (30) in the x-chart, say. Therefore, the Picard-Lindelöf uniqueness theorem gives they must agree on $I_1 \cap I_2 \ni t_0$.

• Extension to a maximal solution: We can uniquely extend our two solutions c_1, c_2 of the previous step to a common solution c defined on $I = I_1 \cup I_2$ and taking values in the union $U \cup V$. Thus we may consider a maximal element $c\colon I \to M$, solving (28) and extending a local solution. We need to show its interval of definition I cannot be extended. So let $I = (a, b)$ where $a \in \{-\infty\} \cup \mathbb{R}$ and $b \in \mathbb{R} \cup \{\infty\}$, and one of these values, say b, is not infinite. Using the fact that M has a compact exhaustion, we can say: In case $c(t_i)$ leaves every compact subset of M for $t_i \nearrow b$, then clearly c does not extend to b.

In the other case, $c(t_i)$ has a subsequence converging to some $q_0 \in K$ where K is a compact subset of M. Pick a chart (x, U_0) around q_0. The Picard-Lindelöf Theorem gives a neighbourhood $U \subset U_0$ of q_0 and a uniform $\varepsilon > 0$, such that for each point $q \in U$ the field X has an integral curve $\tilde{c}\colon (T-\varepsilon, T+\varepsilon)$ through $q = \tilde{c}(T)$, where T is yet to be specified. (See, for instance, Spivak I, 2. Theorem in Chapter 5, to verify that indeed ε can be chosen uniformly.) Now pick a point $c(t_i) \in U$ such that $t_i + \varepsilon > b$ and consider specifically $\tilde{c}$ through $c(t_i)$ taken with $T := t_i$. Then $\tilde{c}$ extends c to times $(a, t_i + \varepsilon)$. Indeed, by the uniqueness theorem this extension is uniquely defined and an integral curve of X. This contradicts the fact that c cannot be extended to time b. $\square$

5. COMMUTING FLOWS AND THE LIE BRACKET

A single vector field has one-dimensional integral curves. In this section, we deal with a higher dimensional generalization: Suppose we have two or more vector fields:
• Can we integrate the fields to a parameterized surface or submanifold such that they become its coordinate vector fields (i.e., standard vector fields)?
• More generally, can we find a surface or integral submanifold which is the linear hull of the given fields?

The answer is non-trivial and based on the notion of the Lie bracket.

5.1. **Flows.** In a stationary moving fluid, the position of a particle p after time t defines a map $\varphi(t,p)$. Differentiation of the flow defines a velocity field $\frac{d}{dt}\varphi(t,p) = X(\varphi(t,p))$. Conversely, the collection of integral curves of a given vector field X as provided by Theorem 21 give a flow $t \mapsto \varphi(t,p)$ of the particles p. Since it is delicate to say for which times the flow is actually defined, let us start by introducing a notion of flow without refering to any ODE:

Definition. A (*local*) *flow [Fluss]* on a manifold M is a differentiable mapping

$$\varphi\colon D \subset \mathbb{R} \times M \to M, \qquad (t,p) \mapsto \varphi_t(p) = \varphi(t,p),$$

where D is subject to the conditions
• D is open,
• $\{0\} \times M \subset D$,
• for all $p \in M$ the set $D \cap (\mathbb{R} \times \{p\})$ is an open interval,
and moreover φ satisfies

$$\varphi_0 = \mathrm{id} \qquad \text{and} \qquad \varphi_{s+t} = \varphi_s \circ \varphi_t \quad \text{whenever defined.} \tag{31}$$

If φ_t is defined on $(a,b) \times M$ it is also called a *local 1-parameter group*. We need some more terminology:

• We call a flow $\varphi\colon D \to M$ *maximal* if φ does not admit an extension to any proper open superset $D' \supset D$ that satisfies the above conditions.
• φ is *global* if φ is defined on all of $\mathbb{R} \times M$.
If M is compact then a maximal flow φ can be shown to be global ($\rightsquigarrow$ problems).

A velocity vector field X can always be integrated to a flow:

Theorem 22. *Given $X \in \mathcal{V}(M)$ there is a unique maximal flow $\varphi\colon D \to M$, such that*

$$\frac{d}{dt}\varphi_t(p) = X\bigl(\varphi_t(p)\bigr) \qquad \text{for all } p \in M. \tag{32}$$

Moreover, if φ is global each φ_t is a diffeomorphism of M.

Proof. Set $\varphi_t(p) := c(t)$ where c is the unique maximal solution of the initial value problem (28) established in Theorem 21. By ODE theory we have the following properties:
• Continuous dependence on initial conditions implies D is open.
• For X differentiable, the ODE solutions depend differentiably on initial conditions, implying that $(t, p) \mapsto \varphi_t(p)$ is differentiable.

To verify $\varphi_{t+s} = \varphi_t \circ \varphi_s$ we claim that $s \mapsto \varphi_{t+s}(p)$ is the integral curve of X through $q := \varphi_t(p)$. Indeed $\varphi_{t+0}(p) = q$ and, by the standard chain rule,

$$\frac{d}{ds}\varphi(t+s, p) = \partial_1\varphi(t+s, p) \cdot \frac{d}{ds}(t+s) = X\bigl(\varphi(t+s, p)\bigr).$$

Let us prove φ_t is a diffeomorphism. By assumption φ_{-t} exists and is differentiable, as pointed out before. Therefore, $\varphi_t \circ \varphi_{-t} = \varphi_{t-t} = \mathrm{id}$, and likewise $\varphi_{-t} \circ \varphi_t = \mathrm{id}$ on all of M. Thus φ_t has the inverse φ_{-t}. ☐

Examples. 1. The field $e_i \in \mathcal{V}(\mathbb{R}^n)$ defines the global flow $\varphi_t(p) = p + te_i$. However, if we remove a point from $\mathbb{R}^n$ the flow will no longer be global.
2. For $M = \mathbb{R}^2$ consider the vertical field $X(u, v) := (0, u)$. Then $\varphi_t(u, v) = (u, v + ut)$. Indeed,

$$\varphi_0(u, v) = (u, v) \quad \text{and} \quad \frac{d}{dt}\varphi_t(u, v) = \frac{d}{dt}(u, v + ut) = (0, u) = X\bigl(\varphi_t(u, v)\bigr).$$

3. The 90 degree rotation field $J(u, v) := (-v, u)$ on $\mathbb{R}^2$ has circles as integral curves and $\varphi_t \in \mathsf{SO}(2)$ is a rotation by an angle t (verify!).
4. For $X \in \mathcal{V}(\mathbb{R}^n)$, we have the expansion $\varphi_t(p) = p + tX(p) + O(t^2)$ at $t = 0$ (problems).

5.2. **The Lie bracket of vector fields.** For $f \in C^\infty(M)$ the Lie derivative $\partial_X f$ is again in $C^\infty(M)$. Thus we can iterate Lie derivatives. Let us compute the second Lie derivative $\partial_X(\partial_Y f) \in C^\infty(M)$ locally in a chart (x,U) where $X = \sum_i \xi^i e_i$ and $Y = \sum_j \eta^j e_j$:

$$\partial_X(\partial_Y f) = \partial_X\Big(\sum_j \eta^j \frac{\partial f}{\partial x^j}\Big) = \sum_{i,j} \xi^i \frac{\partial \eta^j}{\partial x^i}\frac{\partial f}{\partial x^j} + \sum_{i,j} \xi^i \eta^j \frac{\partial^2 f}{\partial x^i \partial x^j} \tag{33}$$

It is no surprise that $\partial_X\partial_Y$ involves second derivatives. However, when we subtract $\partial_Y\partial_X$ from it, the second derivatives cancel. Consequently, locally we are left with first order derivatives at most. This happens to be a global fact:

Theorem 23. *Let X and Y be vector fields on a manifold M. Then there is a unique vector field $Z \in \mathcal{V}(M)$ such that $\partial_Z f = (\partial_X\partial_Y - \partial_Y\partial_X)f$ holds for all $f \in C^\infty(M)$.*

Proof. Using (33) and the Schwarz theorem that second partials commute we obtain, with respect to a chart (x,U),

$$\partial_X\partial_Y f - \partial_Y\partial_X f = \sum_{i,k}\left(\xi^i \frac{\partial \eta^k}{\partial x^i} - \eta^i \frac{\partial \xi^k}{\partial x^i}\right)\frac{\partial f}{\partial x^k} \qquad \text{for all } p \in U \text{ and } f \in C^\infty(M). \tag{34}$$

So on U our claim holds for

$$Z(p) := \sum_k \zeta^k(p) e_k(p) \quad \text{ with principal part } \zeta^k := \sum_i \Big(\xi^i \frac{\partial \eta^k}{\partial x^i} - \eta^i \frac{\partial \xi^k}{\partial x^i}\Big).$$

But for each f, the iterated Lie derivative $(\partial_X\partial_Y - \partial_Y\partial_X)f$ is defined independently of the charts chosen, and so in fact $\partial_Z f$ is defined globally. Thus (34) does not depend on the chart (x,U), and Z is a (global) vector field on M. (You might as well convince yourself that the coefficients ζ^k transform with the Jacobian of the transition map – please check!). $\square$

We write $[X,Y]$ for the vector field Z and call it the *Lie bracket [Lie-Klammer]* or the *commutator [Kommutator]* of X and Y. Then

$$\partial_{[X,Y]} := \partial_X\partial_Y - \partial_Y\partial_X. \tag{35}$$

Examples. 1. Consider a chart (x,U). The standard basis (e_i) defines vector fields with constant principal parts $\xi^j = \delta^i_j$. Hence their commutator vanishes: $\big[e_i, e_j\big] = 0$. This is globally true for Euclidean space.

2. For $M = \mathbb{R}^2$ we identify tangent vectors with principal parts. Let us consider the two fields

$$X(u,v) := (0,u) = \big(0, \xi^2(u,v)\big) \qquad Y(u,v) := (2,0) = \big(\eta^1(u,v), 0\big). \tag{36}$$

The only non-vanishing partial of the principal parts ξ^i, η^i is $\frac{\partial}{\partial u}\xi^2 = 1$. Hence

$$[X,Y] = \big(-\eta^1 \frac{\partial}{\partial u}\xi^2\big)e_2 = (-2 \cdot 1)\, e_2 = -(0,2).$$

The Lie bracket is in fact a more general concept, encountered in various other settings:

Definition. A *Lie algebra* is an $\mathbb{R}$-vector space $\mathcal{A}$ with an $\mathbb{R}$-bilinear map $[\cdot,\cdot]\colon \mathcal{A}\times\mathcal{A} \to \mathcal{A}$ such that for all $X, Y, Z \in \mathcal{A}$ the following holds:
(i) Anti-commutativity $[X,Y] = -[Y,X]$,
(ii) *Jacobi identity* $\big[[X,Y],Z\big] + \big[[Y,Z],X\big] + \big[[Z,X],Y\big] = 0$.

Examples. 1. For $\dim \mathcal{A} = 1$ the Lie bracket must vanish, for $\dim \mathcal{A} = 2$ there are only two Lie brackets up to isomorphism. (See problems)
2. $\mathbb{R}^3$ with the cross product,
3. $n \times n$ matrices with $[A,B] := AB - BA$.
4. (trivial): Any vector space with $[v,w] := 0$.
5. A *Lie group* M is a manifold with a continuous group structure, or a topological group. Let $e \in M$ be the group identity. Then the tangent space at this element, T_eM, has the structure of a Lie algebra. For instance $\mathsf{SO}(n)$ is a Lie group. If E is the identity matrix then $T_E\mathsf{SO}(n) = \{\text{skew symmetric matrices}\}$ is a Lie algebra, where $[.,.]$ is defined as in 3.

We can now state for our Lie bracket:

Theorem 24. (i) *The vector fields* $\mathcal{V}(M)$ *with* $[.,.]$ *defined by* (35) *form a Lie algebra.*
(ii) *The Lie bracket on* $\mathcal{V}(M)$ *satisfies*

(37) $[fX, gY] = fg[X,Y] + f(\partial_X g)Y - g(\partial_Y f)X$ *for all* $f, g \in C^\infty(M),\ X, Y \in \mathcal{V}(M)$.

To verify the Jacobi identity for (i) add the three cyclic permutations of $\big[[X,Y],Z\big] = [XY - YX, Z] = XYZ - YXZ - ZXY + ZYX$, where for simplicity we skipped all Lie derivative symbols ∂. We leave the proof of (37) as an exercise.

5.3. **Commuting flows.** Suppose X, Y are two vector fields with flows φ_s, ψ_t, respectively. It is natural to ask whether $\varphi_s\psi_t(p) = \psi_t\varphi_s(p)$ holds, or equivalently if $\psi_{-t}\varphi_{-s}\psi_t\varphi_s = \mathrm{id}$. It will turn out that $[X,Y]_p = 0$ is an infinitesimal version of these equations.

Examples. 1. Consider the two fields e_1, e_2 in $\mathbb{R}^2$ and the initial point 0. Let φ be the flow of e_1, and ψ be the flow of e_2. Then $\psi_t\varphi_s(0) = (s,t)$ and also $\varphi_s\psi_t(0) = (s,t)$.

2. For example 2. from page 34 we have $\varphi_s(u,v) = (u, v + us)$ and $\psi_t(u,v) = (u + 2t, v)$. Hence

$$\psi_1\varphi_1(0,0) = \psi_1(0,0) = (2,0) \qquad \neq \qquad \varphi_1\psi_1(0,0) = \varphi_1(2,0) = (2,2).$$

We claim that flows commute if their generating vector fields have a vanishing Lie bracket.

Theorem 25. *Let $X, Y \in \mathcal{V}(M)$ be vector fields with flows φ, ψ, respectively. Then $\varphi_s\psi_t(p) = \psi_t\varphi_s(p)$ holds for all $p \in M$ and those s, t for which the equation is defined, if and only if $[X, Y] \equiv 0$.*

Thus the coordinate fields e_i, e_j on a manifold must have vanishing Lie bracket (the converse is true only locally). We need two lemmas for the proof.

Lemma 26. *Let X be a vector field on M and $p \in M$. If $X(p) \neq 0$, then there exists a chart (x, U) around p such that $X = e_1$ on U.*

Proof. Let (y, V) be a chart of M with $y(p) = 0$. Composing y with a rotation and dilation, we may assume the principal part of $X(p)$ points into the direction of the first basis vector of (y, V), meaning that $\xi(p) = b_1$. Let $\{b_1\}^\perp$ be the coordinate hyperplane. Then the restriction of y to the $n-1$-dimensional submanifold $H := y^{-1}\big(\{b_1\}^\perp \cap y(V)\big)$ remains a diffeomorphism.

Let $\varphi\colon D \subset \mathbb{R} \times M \to M$ be the flow of X. Restrict φ to $D \cap (\mathbb{R} \times H)$, and denote this map again by φ. Then:
- $\varphi_0 = \mathrm{id}\,|_H$ and so $d\varphi_{(0,p)}(0, v) = v$, meaning that tangent vectors v to H are preserved.
- $d\varphi_{(0,p)}(e_t, 0) = X_p \notin T_pH$ for e_t the unit vector in time.

Thus the image of $d\varphi_{(0,p)}$ has dimension $(n-1)+1$, that is, $d\varphi$ has rank n at $(0, p)$. From the inverse mapping theorem we conclude φ is a local diffeomorphism on some neighbourhood W of $(0, p) \in \mathbb{R} \times H$ to $U := \varphi(W) \subset M$.

Let us now define x in terms of φ and y: Since φ maps W to U and y maps H to $\{b_1\}^\perp$ we set

$$x\colon U \to \mathbb{R} \times y(H) \subset \mathbb{R}^n, \qquad x := (\mathrm{id}_\mathbb{R}, y) \circ \varphi^{-1};$$

The inverse then is $x^{-1} = \big((\mathrm{id}_\mathbb{R}, y) \circ \varphi^{-1}\big)^{-1} = \varphi \circ (\mathrm{id}_\mathbb{R}, y^{-1})$.

In order to prove the claim, it remains to be shown that the tangents

$$[t \mapsto x^{-1}(u + tb_1)] = \big[t \mapsto \varphi\big(t + u_1, y^{-1}(u_2, \dots, u_n)\big)\big] = \big[t \mapsto \varphi_t\big(\varphi_{u_1}(y^{-1}(u_2, \dots, u_n))\big)\big]$$

agree with X. Indeed they do since φ is the flow of X. □

Problem: Determine the map x for the rotation field $J(u, v) = (-v, u)$ on $\mathbb{R}^2 \setminus \{0\}$.

A vector field on $\mathbb{R}^n$ is a mapping $Y = (Y^1, \ldots, Y^n)\colon \mathbb{R}^n \to \mathbb{R}^n$. The directional derivative of Y in direction of any other vector field X can be expressed in terms of the flow φ of X:

$$\begin{aligned}\partial_X Y(p) &= \lim_{t\to 0} \frac{Y\big(p + tX(p))\big) - Y(p)}{t} = \lim_{t\to 0} \frac{Y\big(\varphi_t(p)\big) - Y(p)}{t} \\ &= \frac{d}{dt}(Y \circ \varphi_t)\Big|_{t=0} = \sum \frac{d(Y\circ\varphi_t)^i}{dt}\Big|_{t=0} e_i\,,\end{aligned} \tag{38}$$

using the fact $\varphi_t = \mathrm{id} + tX + O(t^2)$. On a manifold M, the same difference quotient is no longer meaningful: The vectors $Y\big(\varphi_t(p)\big) \in T_{\varphi_t(p)}M$ and $Y(p) \in T_pM$ are contained in different tangent spaces and so subtraction cannot be defined.

In order to use vector space operations on T_pM alone we use φ^{-1} to move the first vector back to T_pM. Then we can assert:

Lemma 27. *If X, Y are vector fields on M, and φ is the flow of X, then*

$$[X,Y](p) = \lim_{t\to 0} \frac{d\varphi_{-t} Y(\varphi_t(p)) - Y(p)}{t} \qquad \text{for all } p \in M. \tag{39}$$

Problem: Confirm this formula for Example 2 on page 34.

Proof. Let us first consider the case $X(p) \neq 0$. According to Lemma 26 there is a chart (x, U) of M with $X = e_1$. With respect to this chart we have a constant local representation $X(q) = e_1(q)$ for $q \in U$, and so $\partial_{[X,Y]} = \partial_X \partial_Y$ holds in view of (34); here the right hand side is defined only in local coordinates.

The local representation of Y w.r.t. x is $Y(p) = \sum \eta^i(p) e_i(p)$. As long as defined, this gives $\varphi_t\big(x^{-1}(u)\big) = x^{-1}(u_1 + t, u_2, \ldots, u_n)$, and thus $d\varphi_t(e_i(p)) = e_i(\varphi_t(p))$ for all t and $1 \leq i \leq n$. Consequently,

$$F\big(\varphi_t(p)\big) := d\varphi_{-t} Y(\varphi_t(p)) = \sum_i \eta^i\big(\varphi_t(p)\big) e_i(p).$$

The difference quotient in T_pM then verifies (39):

$$\begin{aligned}\lim_{t\to 0} \frac{d\varphi_{-t} Y_{\varphi_t(p)} - Y_p}{t} &= \lim_{t\to 0} \frac{F\big(\varphi_t(p)\big) - F(p)}{t} \overset{(38)}{=} \sum_i \frac{d}{dt} \eta^i\big(\varphi_t(p)\big)\Big|_{t=0} e_i(p) \\ &= \sum_i \partial_X \eta^i(p) e_i(p) = \sum_i (\partial_X \eta^i(p) - \partial_Y \xi^i(p)) e_i(p) = [X,Y](p)\end{aligned}$$

In case $X(p)$ vanishes identically on a neighbourhood of p, then $[X,Y](p) = 0$ on the one hand, and $\varphi_t = \mathrm{id}$ on the other hand, and so the right hand side of (39) vanishes. Finally, the case $X(p) = 0$, but $X(p_k) \neq 0$ for a sequence $p_k \to p$ results from the first case, by considering $k \to \infty$ and using continuity of our local representation of the difference quotient in p. $\square$

Proof of Thm. 25. "$\Longrightarrow$" Given the commuting property of the flows, the previous lemma saves us from the need to differentiate the flow equation twice. Indeed,

$$\begin{aligned}(d\varphi_{-t})_p Y\big(\varphi_t(p)\big) - Y(p) &= d\varphi_{-t}\frac{d}{ds}\psi_s\big(\varphi_t(p)\big)\Big|_{s=0} - \frac{d}{ds}\psi_s(p)\Big|_{s=0} \\ &\overset{\text{chain rule}}{=} \frac{d}{ds}\Big[\big(\varphi_{-t}\circ\psi_s\circ\varphi_t - \psi_s\big)(p)\Big]_{s=0}\end{aligned} \tag{40}$$

which vanishes by assumption. Hence $[X,Y]=0$ by Lemma 27.

"$\Longleftarrow$" Here we must integrate our condition on the vector fields. Instead, we appeal to the uniqueness assertion of the Picard-Lindelöf theorem.

Let $Z(t) := d\varphi_{-t}Y(\varphi_t(p))$. We prove $\frac{d}{dt}Z(t)=0$ for t small.

$$\begin{aligned}\frac{d}{d\tau}Z(t+\tau)\Big|_{\tau=0} &= \frac{d}{d\tau}d\varphi_{-t-\tau}Y(\varphi_{t+\tau}(p))\Big|_{\tau=0} = \frac{d}{d\tau}d\varphi_{-t}d\varphi_{-\tau}Y\big(\varphi_{\tau+t}(p)\big)\Big|_{\tau=0} \\ &\overset{\text{chain rule}}{=} d^2\varphi_{-t}\underbrace{\left(\frac{d}{d\tau}(d\varphi_{-\tau}\circ Y\circ\varphi_\tau)\Big|_{\tau=0}\right)}_{=0 \text{ by ass. \& Lemma 27}}\circ\varphi_t(p) = 0\end{aligned} \tag{41}$$

To see the last equality sign holds, note that the differential $d^2\varphi_{-t}$ of the linear map $d\varphi_{-t}$ is again the same linear map, and so maps 0 to 0. From (41) we conclude that $Z(t)$ must be constant. Thus $Z(0)=Y(p)$ equals $Z(t)=d\varphi_{-t}Y(\varphi_t(p))$.

It follows from (40) that for fixed t, the vector field $d\varphi_{-t}Y(\varphi_t(p))$ has the flow $s\mapsto\varphi_{-t}\circ\psi_s\circ\varphi_t$. Together with the last results this gives that Y has the flow ψ_s as well as the flow $s\mapsto\varphi_t\circ\psi_s\circ\varphi_{-t}$. But the local flow is unique, and so $\psi_s=\varphi_t\circ\psi_s\circ\varphi_{-t}$ which is the claim. ☐

5.4. **Frobenius theorem.** We now generalize integral curves to integral surfaces or manifolds:

Definition. (i) An n-dimensional *distribution* Δ on a manifold M^{n+k} is a mapping $p\mapsto\Delta(p)\subset T_pM$, where $\Delta(p)$ is an n-dimensional subspace. Here, the assignment must be smooth in the sense that each point p has a neighbourhood U and n vectorfields $X_1,\dots,X_n$ exist on U which span Δ at each point $p\in U$.
(ii) An n-dimensional submanifold $N\subset M$ is called an *integral manifold* of Δ if the inclusion map $i\colon N\to M$ satisfies $di_p(T_pN)=\Delta(p)$.
(iii) An n-dimensional distribution Δ is called *(locally) integrable*, if each $p\in M$ is contained in an integral submanifold of Δ.

Examples. 1. A nonvanishing vector field defines a one-dimensional distribution. The integral manifolds are integral curves, so a one-dimensional distribution is always integrable.
2. On the torus $\mathbb{R}^2/\mathbb{Z}^2$, a constant vector field with irrational slope defines a one-dimensional

distribution. It is locally integrable: The integral manifolds are lines of irrational slope. However, globally the infinite irrational line is not a submanifold. That is, there may not be a maximal integral submanifold.

3. Integral submanifolds need not exist, not even locally. A simple example is a 2-plane distribution in $\mathbb{R}^3$, spanned by $X(p) := e_1$ and $Y(p) := e_2 + p^1 e_3$.

We will relate integrability to the following.

Definition. A distribution Δ is *involutive* if for $X, Y \in \mathcal{V}(M)$ such that $X(p), Y(p) \in \Delta(p)$ for all $p \in M$ also $[X, Y](p) \in \Delta(p)$.

Example. For $X(p) := e_1$ and $Y(p) := e_2 + p^1 e_3$ we have $[X, Y] = e_3$, and so the distribution $\Delta(p) = \operatorname{span}\{X(p), Y(p)\}$ is not involutive.

It is enough to check involutiveness on a basis:

Lemma 28. *Suppose each $p \in M^{n+k}$ has a neighbourhood U such that $X_1, \ldots, X_n \in \mathcal{V}(U)$ span Δ and $[X_i, X_j](p) \in \Delta(p)$ for all $1 \le i, j \le n$ then Δ is involutive.*

Proof. This is a linear algebra fact: If $X = \sum \xi^i X_i$ and $Y = \sum \eta^j X_j$ then indeed

$$[X, Y] = \sum_{i,j} [\xi^i X_i, \eta^j X_j] \overset{(37)}{=} \sum_{i,j} \Big(\xi^i \eta^j [X_i, X_j] + \xi^i \partial_{X_i} \eta^j X_j - \eta^j \partial_{X_j} \xi^i X_i \Big) \in \Delta \qquad \square$$

We need some preparatory notions and lemmas. To calculate Lie brackets, it is useful to know how the Lie bracket transforms under a differentiable map $\varphi \colon M \to \tilde{M}$. We call $X \in \mathcal{V}(M)$ and $\tilde{X} \in \mathcal{V}(\tilde{M})$ *φ-related [φ-verwandt]*, if

$$d\varphi(X) \equiv \tilde{X} \circ \varphi.$$

Note that the integral curves of φ-related vector fields are related as images under φ.

Suppose a curve $c(t)$ represents X at p, that is, $X(p) = [c]$. Then Lie derivatives are easily related, namely for all $f \in C^\infty(\tilde{M})$

$$\begin{aligned}(\partial_{\tilde{X}} f)(\varphi(p)) = \partial_{\tilde{X}(\varphi(p))} f = \partial_{d\varphi[c]} f &\overset{\text{def. differential}}{=} \partial_{[\varphi \circ c]} f \\ &\overset{\text{def. Lie der.}}{=} \frac{d}{dt}(f \circ \varphi \circ c)(0) \overset{\text{def. Lie der.}}{=} \partial_{[c]}(f \circ \varphi) = \partial_X (f \circ \varphi)(p)\end{aligned}$$

and so

$$(\partial_{\tilde{X}} f) \circ \varphi = \partial_X (f \circ \varphi). \tag{42}$$

The following result is no surprise in view of the fact that the Lie bracket measures the extent to which the flows of two vector fields commute:

Lemma 29. *Suppose $X, Y \in \mathcal{V}(M)$ are φ-related to $\tilde{X}, \tilde{Y} \in \mathcal{V}(\tilde{M})$. Then $[X, Y]$ is φ-related to $[\tilde{X}, \tilde{Y}]$, that is, $[\tilde{X}, \tilde{Y}](\varphi(p)) = d\varphi_p[X, Y] =: \widetilde{[X, Y]}(p)$.*

Proof. For any $f \in C^\infty(\tilde{M})$ we have:

$$\begin{aligned}(\partial_{[\tilde{X},\tilde{Y}]} f) \circ \varphi &= \big(\partial_{\tilde{X}}(\partial_{\tilde{Y}} f)\big) \circ \varphi - \big(\partial_{\tilde{Y}}(\partial_{\tilde{X}} f)\big) \circ \varphi \\ &\overset{(42)}{=} \partial_X\big((\partial_{\tilde{Y}} f) \circ \varphi\big) - \partial_Y\big((\partial_{\tilde{X}} f) \circ \varphi\big) \overset{(42)}{=} \partial_X\big(\partial_Y(f \circ \varphi)\big) - \partial_Y\big(\partial_X(f \circ \varphi)\big) \\ &= \partial_{[X,Y]}(f \circ \varphi)\end{aligned}$$

Consequently

$$df[\tilde{X}, \tilde{Y}] \circ \varphi = df(d\varphi[X, Y])$$

which means, as desired, $[\tilde{X}, \tilde{Y}] \circ \varphi = d\varphi[X, Y]$. □

We now generalize Lemma 26 to several vector fields.

Proposition 30. *Let $X_1, \dots, X_n$ be linearly independent vector fields on a $n+k$-dimensional manifold M^{n+k}, defined in a neighbourhood of a point p. Suppose that on this neighbourhood, $[X_i, X_j] \equiv 0$ for $1 \le i, j \le n$. Then there is a coordinate system (x, U) around p with standard basis e_j, such that $X_i = e_i$ on U for $i = 1, \dots, n$.*

Proof. We may choose a chart (y, V) such that $y(p) = 0$, and that $X_j(p)$ has the j-th basis vector b_j of $\mathbb{R}^{n+k}$ as its principal part. Each flow φ^j generated by X_j is defined in a neighbourhood of $\{0\} \times M$. Thus there is a neighbourhood $U(0)$ of 0 such that

$$\begin{aligned}&\chi\colon U(0) \subset \mathbb{R}^{n+k} \to M, \\ &\chi(u^1, \dots, u^{n+k}) := \varphi^1_{u^1}\Big(\varphi^2_{u^2}\Big(\cdots\Big(\varphi^n_{u^n}\big(0, \dots, 0, u^{n+1}, \dots, u^{n+k}\big)\Big)\cdots\Big)\Big),\end{aligned}$$

is defined. Then $d\chi_0(b_i) = \mathrm{id}$ since

$$d\chi_0(b_i) = \begin{cases} X_i(0) = b_i & i = 1, \dots, k, \\ b_i & i = n+1, \dots, n+k. \end{cases}$$

Hence χ is a diffeomorphism in a neighbourhood of $0 \in \mathbb{R}^{n+k}$, and we may define $x := \chi^{-1}$ as a chart in some neighbourhood U of $p = 0$.

We have $X_1 = e_1$ since the curves $u + tb_1$ have tangent vector X_1. We now use our hypothesis on the Lie bracket to prove the same for the indices from 2 to n. By Thm. 25 the hypothesis allows us to write

$$\chi(u^1, \dots, u^{n+k}) := \varphi^j_{u^j}\Big(\varphi^1_{u^1}\Big(\cdots\Big(\varphi^n_{u^n}\big(0, \dots, 0, u^{n+1}, \dots, u^{n+k}\big)\Big)\cdots\Big)\Big),$$

and so as before we have $X_j = e_j$ for $j = 1, \dots, n$ as well. □

Theorem 31. *An involutive n-dimensional distribution Δ on a manifold M^{n+k} is integrable. More precisely, for each $p \in M^{n+k}$ there is a chart (x, U) such that for each $a \in \mathbb{R}^k$ the set $N(p, a) := \{q \in U : (x^{n+1}(q), \dots x^{n+k}(q)) = a\}$ is an integral submanifold with $p \in N(p, 0)$.*

Proof. Let us first prove the theorem for $M = \mathbb{R}^{n+k}$. By a motion, we can assume $p = 0$ and $\Delta(0) = \mathbb{R}^n \times \{0\}$. Denote with $\pi = d\pi \colon \mathbb{R}^{n+k} \to \mathbb{R}^n$ the projection onto the first n components. Since $d\pi_0$ restricted to $\Delta(0)$ is an isomorphism to $\mathbb{R}^n$, by continuity there is a neighbourhood U of 0, such that the restriction $d\pi \colon \Delta(q) \to \mathbb{R}^n$ is bijective for all $q \in U$. Hence the preimage of the standard basis defines vector fields $X_1, \dots, X_n \in \Delta(q)$ with $d\pi(X_i) = e_i$ for $i = 1, \dots, n$. That is, the vector fields e_i and X_i are π-related. By Lemma 29,

$$d\pi([X_i, X_j](q)) = [e_i, e_j](\pi(q)) = 0.$$

Since $d\pi$ is an isomorphism on $\Delta(q) = \operatorname{span}\{X_1(q), \dots, X_n(q)\}$ this implies $[X_i, X_j] = 0$ for all $i, j \leq n$.

Hence we can apply Proposition 30 to obtain a coordinate system (y, U) such that the X_i become the standard basis. Then for each $a \in \mathbb{R}^k$, the sets $\{q \in U : y^{n+1} = a^1, \dots, y^{n+k} = a^k\}$ are integral manifolds.

To obtain the result for a manifold, consider an arbitrary chart (x, U). On the chart image $x(U)$, apply the above considerations, and map the integral manifolds obtained in $x(U)$ back to the manifold by x^{-1}. $\square$

Part 3. Differential forms and Stokes' theorem

Stokes' theorem generalizes the fundamental theorem of calculus to several dimensions, in a way to include all the classical integral theorems like the divergence theorem or Green's theorem. To set up a formalism for generalizing these theorems we proceed as follows:
1. We define k-dimensional area elements for $\mathbb{R}^n$ and
2. we let these area elements depend on footpoints of a manifold ("differential forms").
3. We define integration of differential forms over a manifold.

All this will be done taking orientation into account; unlike surface integrals in terms of the Gram determinant, our integral will be orientation dependent.

I prepared this part using Spivak's book [Spi]. I recommend Agricola/Friedrich [AF] as a more modern reference and for the wide range of applications of forms in geometry and physics. It is also worth to compare our presentation with a source which presents the theorem for submanifolds of $\mathbb{R}^n$, such as Forster's Analysis 3: Surprisingly, the amount of technical work saved there is marginal.

6. DIFFERENTIAL FORMS

6.1. **Multilinear algebra.** For a motivation consider area and volume. The signed area of a rectangle spanned by v, w in $\mathbb{R}^2$ only depends on $v \times w = -w \times v$, that is, on an alternating 2-form of v, w. Similarly the signed volume of a parallelepipied spanned by u, v, w in $\mathbb{R}^3$ is $\det(u, v, w) = \langle u \times v, w \rangle$, which is also alternating in its three entries.

To generalize this theory to k-dimensional area elements in $\mathbb{R}^n$, consider a real vector space V of dimension n, with dual space V^*.

Definition. (i) A function $T\colon V^k \to \mathbb{R}$ is *k-multilinear* or a *k-tensor* if

$$v_i \mapsto T(v_1, \ldots, v_i, \ldots, v_k)$$

is linear for each i and all $v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_k \in V$.
(ii) A k-tensor T is *alternating*, or a *k-form*, if for all $v_1, \ldots, v_k \in V$

$$T(v_1, \ldots, v_i, \ldots, v_j, \ldots, v_k) = -T(v_1, \ldots, v_j, \ldots, v_i, \ldots, v_k) \quad \forall\, 1 \le i \ne j \le k.$$

(iii) We denote the vector space of k-tensors by $\otimes^k V$, and the subspace of k-forms by $\Lambda^k V$.

We also define 0-tensors to be real numbers, that is, $\otimes^0 V = \Lambda^0 V := \mathbb{R}$.

Examples. 1. $\otimes^1 V = \Lambda^1 V = V^*$,
2. $\otimes^2 V = \{\text{bilinear maps on } V\}$ (called bilinear forms in Linear Algebra).
3. $\det \in \Lambda^n V$.

That a tensor T is alternating is equivalent to any of the following properties:
• T vanishes if any pair of vectors coincides, $v_i = v_j$ for $i \neq j$ (use polarisation).
• $T\big(v_{\sigma(1)}, \dots, v_{\sigma(k)}\big) = (\operatorname{sgn}\sigma)\, T(v_1, \dots, v_k)$ for all permutations $\sigma \in S^k$ and all $v_i \in V$.

Definition. The *tensor product* is the map

$$\otimes\colon \otimes^k V \times \otimes^l V \to \otimes^{k+l} V, \qquad \big(S \otimes T\big)(v_1, \dots, v_{k+l}) := S(v_1, \dots, v_k)\, T(v_{k+1}, \dots, v_{k+l}).$$

We seek a similar product for forms. However, if S and T are alternating, the tensor product $S \otimes T$ need not be alternating. For example, suppose $k = l = 1$, that is, $S, T \in V^*$. Then the bilinear form $S \otimes T$ is alternating, if and only if $v \mapsto S(v)T(v)$ vanishes identically. However, $S \wedge T := S \otimes T - T \otimes S$ is alternating as $S \wedge T(v, v) = S(v)T(v) - T(v)S(v) \equiv 0$.

For the general case, we first construct a projection map

$$\operatorname{Alt}\colon \otimes^k V \to \Lambda^k V, \qquad (\operatorname{Alt} T)(v_1, \dots, v_k) := \frac{1}{k!} \sum_{\sigma \in S_k} (\operatorname{sgn}\sigma)\, T\big(v_{\sigma(1)}, \dots, v_{\sigma(k)}\big).$$

Check that $\operatorname{Alt}(T)$ is indeed alternating, and $\operatorname{Alt}(T) = T$ for $T \in \Lambda^k V$ (thereby justifying the factor $1/k!$). We use Alt to define an alternating product of two forms:

Definition. The *wedge* or *exterior product [Dach-/äußeres Produkt]* is the map

$$\wedge\colon \Lambda^k V \times \Lambda^l V \to \Lambda^{k+l} V, \qquad \omega \wedge \eta := \frac{(k+l)!}{k!\, l!} \operatorname{Alt}(\omega \otimes \eta).$$

Example. For $\omega, \eta \in \Lambda^1 V = V^*$ this gives, as before,

$$\omega \wedge \eta\, (v, w) = 2 \operatorname{Alt}(\omega \otimes \eta)(v, w) = \omega \otimes \eta(v, w) - \omega \otimes \eta(w, v) = \big(\omega \otimes \eta - \eta \otimes \omega\big)(v, w). \tag{43}$$

In particular, for our one-forms, $\omega \wedge \eta = -\eta \wedge \omega$, and $\omega \wedge \omega = 0$.

The wedge product has the following properties (problems?):
• Bilinearity: $(\omega, \eta) \mapsto \omega \wedge \eta$ is linear in each argument.
• Anti-commutativity: If $\omega \in \Lambda^k V$ and $\eta \in \Lambda^l V$ then $\omega \wedge \eta = (-1)^{kl} \eta \wedge \omega$. In particular, $\omega \wedge \omega = 0$ for k odd (exhibit, however, an example $\omega \in \Lambda^2 V$ such that $\omega \wedge \omega \neq 0$).
• Associativity: $(\omega \wedge \eta) \wedge \vartheta = \omega \wedge (\eta \wedge \vartheta)$. See [Sp], Thm. 2 of Ch. 7.
• Normalization: If (e_k) is a basis of V and (e^i) the dual basis of V^*, i.e., $e^i(e_k) = \delta^i_k$. then $(e^1 \wedge \ldots \wedge e^n)(v_1, \dots, v_n) = \det(v_1, \dots, v_n)$, justifying the factorial term of the wedge product (see also [Sp], p. 279).

These properties become particularly evident once we exhibit a basis. Perhaps it is useful to exhibit bases for the simple case $k = 2$ first:

Examples. 1. Using again the pair of dual bases (e_k) and (e^i) we obtain

$$(e^i \otimes e^j)\big(\textstyle\sum_k v^k e_k,\, \sum_l w^l e_l\big) = e^i\big(\sum_k v^k e_k\big)\, e^j\big(\sum_l w^l e_l\big) = v^i w^j.$$

Therefore, a bilinear map $b \in \otimes^2 V$ satisfies

$$b(v,w) = \sum_{ij} b(e_i,e_j)\, v^i w^j = \sum_{ij} b(e_i,e_j)\; e^i \otimes e^j\,(v,w).$$

Setting $b_{ij} := b(e_i,e_j)$ we obtain the desired representation: $b = \sum_{ij} b_{ij}\, e^i \otimes e^j$.
2. If, moreover, $b \in \Lambda^2(V)$ is alternating then the coefficients satisfy $b_{ij} = -b_{ji}$ and $b_{ii} = 0$. This yields the identity $\sum_{i>j} b_{ij}\, e^i \otimes e^j = \sum_{j>i} b_{ji}\, e^j \otimes e^i = \sum_{j>i} b_{ij}(-e^j \otimes e^i)$, leading to the representation

$$b = \Big(\sum_{i<j} + \sum_{i>j}\Big)\big(b_{ij}\, e^i \otimes e^j\big) = \sum_{i<j} b_{ij}\big(e^i \otimes e^j - e^j \otimes e^i\big) \overset{(43)}{=} \sum_{i<j} b_{ij}\, e^i \wedge e^j.$$

We now address the general case:

Lemma 32. *Suppose V^* has a basis $e^1,\dots,e^n$. Then*
(i) $\{e^{i_1} \otimes \dots \otimes e^{i_k} : 1 \le i_1,\dots,i_k \le n\}$ is a basis for $\otimes^k V$, and so $\dim \otimes^k V = n^k$*;*
(ii) $\{e^{i_1} \wedge \dots \wedge e^{i_k} : 1 \le i_1 < \dots < i_k \le n\}$ is a basis for $\Lambda^k V$, and so $\dim \Lambda^k V = \binom{n}{k}$.

Proof. It is straightforward to check (i), so we prove (ii). Let (e^i) be the dual basis and choose a multiindex $I = (i_1,\dots,i_k)$. Using the stated properties and (43) we see that if I contains a pair of coinciding indices then $e^I = 0$. Hence we may assume that all indices in I are pairwise distinct. Now pick another multiindex $J = (j_1,\dots,j_k)$ and compute for arbitrary $v_{j_k} \in V$

$$\begin{aligned} e^{i_1} \wedge \dots \wedge e^{i_k}\big(v_{j_1},\dots,v_{j_k}\big) &= k!\,\mathrm{Alt}\,\big(e^{i_1} \otimes \dots \otimes e^{i_k}\big)\big(v_{j_1},\dots,v_{j_k}\big) \\ &= \sum_{\sigma \in S_k} (\operatorname{sgn}\sigma)\big(e^{i_1} \otimes \dots \otimes e^{i_k}\big)\big(v_{\sigma(j_1)},\dots,v_{\sigma(j_k)}\big) = \sum_{\sigma \in S_k} (\operatorname{sgn}\sigma)\, e^{i_1}(v_{\sigma(j_1)}) \dots e^{i_k}(v_{\sigma(j_k)}). \end{aligned}$$

In particular, replacing the v's by basis elements (e_i) of V, we see:

$$\text{either} \quad e^I(e_J) = \operatorname{sgn}(\sigma) \;\text{ if } \exists\, \sigma \in S_k : I = \sigma(J) \qquad \text{or else} \qquad e^I(e_J) = 0.$$

We conclude that $e^I(e_J)$ is nonzero if and only if I and J, considered as sets, have k elements and agree. In particular, the set of vectors in (ii) is linearly independent: We evaluate the linear combination $\sum_{i_1<\dots<i_k} a_{i_1,\dots,i_k} e^{i_1} \wedge \dots \wedge e^{i_k} = 0$ on $e_J = \big(e_{j_1},\dots,e_{j_k}\big)$ (with J an increasing multiindex) to see it implies $a_J = a_{j_1,\dots,j_k} = 0$ for each J.

The n^k vectors $\{e^I\}$ of (i) span $\otimes^k V$, so they also span the subspace $\Lambda^k V \subset \otimes^k V$. To show the subset of vectors with increasing indices I suffice to span $\Lambda^k V$, we use the property that for any given $\tau \in S_k$ we have $S_k = \{\sigma \circ \tau^{-1} : \sigma \in S_k\}$. Hence, by relabelling the permutations:

$$\begin{aligned}\big(e^{\tau(i_1)} \wedge \cdots \wedge e^{\tau(i_k)}\big)\big(v_1, \ldots, v_k\big) &= \sum_{\sigma \in S_k} (\operatorname{sgn} \sigma)\, e^{\tau(i_1)}(v_{\sigma(1)}) \ldots e^{\tau(i_k)}(v_{\sigma(k)}) \\ &= \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma \circ \tau^{-1})(\operatorname{sgn} \tau)\, e^{i_1}(v_{\sigma\circ\tau^{-1}(1)}) \ldots e^{i_k}(v_{\sigma\circ\tau^{-1}(k)}) \qquad (44) \\ &= \operatorname{sgn}(\tau) \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma) e^{i_1}(v_{\sigma(1)}) \ldots e^{i_k}(v_{\sigma(k)}) = \operatorname{sgn}(\tau)\big(e^{i_1} \wedge \cdots \wedge e^{i_k}\big)\big(v_1, \ldots, v_k\big).\end{aligned}$$

□

By anti-commutativity, permuted basis elements differ by at most a sign: $e^2 \wedge e^1 = -e^1 \wedge e^2$ or $e^3 \wedge e^1 \wedge e^2 = -e^1 \wedge e^3 \wedge e^2 = e^1 \wedge e^2 \wedge e^3$.

Examples. 1. For $k > n$, we have $\Lambda^k V = \{0\}$.
2. For $k = n$, we have $\dim \Lambda^n V = 1$. Since $\det \in \Lambda^n V$, any n-form must be a scalar multiple of the determinant.
3. $e^1 \wedge e^2\,(v, w)$ is an area element in the sense that it is the signed area of the projection of the parallelogram (v, w) to the (e_1, e_2)-plane, (see problems).
4. The 2-form $\omega := e^1 \wedge e^2 + e^3 \wedge e^4$ is not a classical area element, but a linear combination of them (compare problems).

Remark. Similarly, k-forms $\Lambda^k \mathbb{R}^n$ can be regarded as the closure of k-dimensional area elements under vector space operations. We will encounter the idea of taking the algebraic closure of geometric objects soon again.

Definition. For $A\colon V \to W$ linear, the map

$$A^*\colon \Lambda^k W \to \Lambda^k V\colon \qquad \omega \mapsto (A^*\omega)(v_1, \ldots, v_k) := \omega(Av_1, \ldots, Av_k)$$

defines the *pull-back [Zurückziehung]* $A^*\omega$ of ω from W to V.

Note that indeed $A^*\omega$ is alternating and multilinear. Moreover, $A^*(\omega \wedge \eta) = A^*\omega \wedge A^*\eta$.

In particular for an endomorphism $A\colon V \to V$ and the determinant $\det \in \Lambda^n V$, the rules of determinant multiplication give

$$(A^* \det)(v_1, \ldots, v_n) = \det(Av_1, \ldots, Av_n) \;=\; \det A\, \det(v_1, \ldots, v_n).$$

Geometrically, this describes the change of signed volume of a parallelepiped under a linear map. Now an arbitrary n-form ω is but a constant multiple of the determinant. Using linearity of $\omega \mapsto A^*\omega$ this gives:

Proposition 33. *Let $\omega \in \Lambda^n V$ for $n = \dim V$ and $A \in \operatorname{End}(V)$. Then*

$$A^*\omega = (\det A)\, \omega. \qquad (45)$$

6.2. **Alternating forms on manifolds.** For the case of a manifold M we take $V := T_pM$ and require differentiable dependence on p, similar to the previous transition from vectors to vector fields.

Definition. Let M^n be a manifold. A family of k-forms $\omega = \{\omega_p \in \Lambda^k T_pM : p \in M\}$ is called a *(differentiable) k-form on M* if the mapping

$$p \mapsto \omega_p\big(X_1(p), \dots, X_k(p)\big) \quad \text{is in } C^\infty(M) \text{ for all } X_1, \dots, X_k \in \mathcal{V}(M).$$

The set of all differentiable k-forms on M is denoted by $\Lambda^k M$; we set $\Lambda^0 M := C^\infty(M)$. The set of all such forms is denoted by $\Lambda M := \bigoplus_{k=0}^n \Lambda^k M$.

Similarly, *tensor fields* and their spaces $\otimes^k M$ or $\bigotimes M = \bigcup_{k\ge 0} \otimes^k M$ can be defined.

Examples. 1. If $f \in C^\infty(M)$ then $df \in \Lambda^1(M)$. Indeed, $df(X) \in C^\infty(M)$ for all $X \in \mathcal{V}(M)$.
2. Consider $\mathbb{R}^n$, with dual basis e^i. Then fe^i is a 1-form for any $f \in C^\infty(\mathbb{R}^n)$. Likewise, if and $e^I = e^{i_1} \wedge \dots \wedge e^{i_k}$ and $f_I \in C^\infty(\mathbb{R}^n)$ then $\sum_{|I|=k} f_I e^I$ is an $|I|$-form.
3. Similarly on the torus $T^n = \mathbb{R}^n/\mathbb{Z}^n$ provided f is $\mathbb{Z}^n$-periodic.
4. We can locally represent a k-form ω in terms of a chart (x, U) by

$$\omega = \sum_{i_1 < \dots < i_k} \omega_{i_1 \dots i_k} e^{i_1} \wedge \dots \wedge e^{i_k} = \sum_{|I|=k} \omega_I e^I \qquad \forall p \in U,$$

with smooth coefficient functions $\omega_I \colon U \to \mathbb{R}$. To obtain forms on M, we can piece them together globally by a partition of unity. Let us note that a traditional notation for standard and dual basis of a chart is

$$\frac{\partial}{\partial x_j} = e_j \quad \text{and} \quad dx^i = e^i \quad \text{satisfying} \quad dx^i\Big(\frac{\partial}{\partial x_j}\Big) = \delta^i_j. \tag{46}$$

The stated properties of k-forms result in the following properties of differential forms:
- k-forms are $C^\infty(M)$-*multilinear*, that is,

$$\omega(X_1, \dots, fX_i, \dots, X_k) = f\omega(X_1, \dots, X_i, \dots, X_k) \quad \text{for all } f \in C^\infty(M) \text{ and } 1 \le i \le k.$$

In particular, this includes the ($\mathbb{R}$-)linearity of $X_i \mapsto \omega(X_1, \dots, X_i, \dots, X_k)$ for each i.
- There is again a wedge product

$$\wedge \colon \Lambda^k M \times \Lambda^l M \to \Lambda^{k+l} M,$$

defined pointwise. It is $C^\infty(M)$-bilinear, i.e., $(f\omega) \wedge \eta = \omega \wedge (f\eta)$, and is anti-commutative in the sense $\omega \wedge \eta = (-1)^{kl} \eta \wedge \omega$, as well as associative. It endows ΛM with the structure of a graded algebra.
- k-forms can be pulled back along $f \colon M \to N$ by pulling them back pointwise with the differential:

$$f^* \colon \Lambda^k N \to \Lambda^k M : \quad (f^*\omega)_p(X_1(p), \dots, X_k(p)) := \omega_{f(p)}\big(df_p(X_1(p)), \dots, df_p(X_k(p))\big)$$

From this definition, $f^*(\omega \wedge \eta) = f^*\omega \wedge f^*\eta$ is obvious. Invoking the chain rule, we arrive at $f^*(g^*(\omega)) = (g \circ f)^*(\omega)$ – note the change in order!

The first and third property hold equally well for tensor fields.

To show below that k-forms have a well-defined integral over manifolds, we will need to consider changes of coordinates. The important case contained of the next statement is that under a change of chart a form transforms precisely with the Jacobian of the transition map, consistent with the change of variables formula:

Theorem 34. *Let $F\colon M^n \to N^n$ be differentiable. Consider charts (x, U) at $p \in M$ with standard basis e^i, as well as (y, V) at $F(p) \in N$ with standard basis f^i. Then*

$$F^*(hf^1 \wedge \cdots \wedge f^n) = (h \circ F) \det \left(\frac{\partial(y^i \circ F)}{\partial x^j}\right)_{ij} e^1 \wedge \cdots \wedge e^n \qquad \text{for all } h \in C^\infty(N).$$

We skip the proof which uses the Jacobian of F in Proposition 33, see [Sp], Ch.7, Thm.7 (by linearity, it is sufficient to consider the case $h \equiv 1$).

Using the notion of orientability, see problems, we assert:

Theorem 35. *On a manifold M^n there exists an n-form ω with $\omega(p) \neq 0$ for all $p \in M$ if and only if M is orientable.*

To construct ω, use an orientable atlas and sum determinants. Indeed, for a partition of unity (φ_α) verify that $\omega := \sum_\alpha \varphi_\alpha x_\alpha^* \det$ cannot vanish pointwise. Conversely, for an atlas, we can use ω to make it orientable, by flipping orientation of charts where $\omega(e_1, \ldots, e_n) < 0$.

6.3. **Differential of a form.** In order to generalize the fundamental theorem $\int_I f' = f|_{\partial I}$ to forms it will be essential to introduce a notion of derivative for forms. For the case of 0-forms, differentiation is already defined in terms of the Lie derivative:

$$d\colon\ \Lambda^0 M = C^\infty(M) \ \to\ \Lambda^1 M = \mathcal{V}^*(M), \qquad f \mapsto df \quad \text{where } df(X) = \partial_X f.$$

Locally, $df = \sum_{j=1}^n \frac{\partial f}{\partial x^j} e^j$. Indeed,

$$\sum_j \frac{\partial f}{\partial x^j} e^j \Big(\sum_i \xi^i e_i\Big) = \sum_j \xi^j \frac{\partial f}{\partial x^j} = df(X).$$

We extend the d-operator to k-forms by applying the differential to the coefficients:

Definition. Let $\omega \in \Lambda^k M$. Then, for each chart (x, U) with standard basis (e^i) we set

$$(47) \quad d\colon \Lambda^k M \to \Lambda^{k+1} M, \qquad \omega = \sum_{|I|=k} \omega_I e^I \mapsto d\omega := \sum_I d\omega_I \wedge e^I = \sum_I \sum_r \frac{\partial \omega_I}{\partial x^r} e^r \wedge e^I.$$

Examples. 1. For the 1-form e^i we have $de^i = d(1e_i) = 0$, likewise for the constant coefficient k-forms e^I.

2. If $\omega \in \Lambda^n M$ then $d\omega = 0$.

3a) On $\mathbb{R}^3$ the 2-form $\omega := f_1\, e^2 \wedge e^3 + f_2\, e^3 \wedge e^1 + f_3\, e^1 \wedge e^2$ has $d\omega = \operatorname{div} f\, e^1 \wedge e^2 \wedge e^3$.

b) Still for $\mathbb{R}^3$, let us set $f := \operatorname{curl} g$ in the previous example, where

$$\operatorname{curl} X = \operatorname{rot} X = (\partial_2 X_3 - \partial_3 X_2,\ \partial_3 X_1 - \partial_1 X_3,\ \partial_1 X_2 - \partial_2 X_1)$$

is the curl or *rotation* of a vector field X. Then the 1-form $\eta := g_1 e^1 + g_2 e^2 + g_3 e^3$ satisfies $d\eta = \omega$ with ω as before. In particular, $d^2\eta = d\omega = \operatorname{div}(\operatorname{curl} g) e^1 \wedge e^2 \wedge e^3 = 0$.

Remark. The classical notation $e^i = dx^i$, see (46), now acquires a further meaning, namely the differential of the function x^i. This is unambiguous on $\mathbb{R}^n$. However, for examples such as the torus T^n, there is no global function x^i with $dx^i = e^i$. To avoid this ambiguity, I refrain from using dx^i, unless this is a true differential.

On a general manifold, we will only show in Prop. 37 that $d\omega$ is well-defined globally, i.e., independent of the choice of chart. Thus for now the following statement applies to a manifold $M = U$ with one chart (x, U); only the next proposition will give it holds for any M.

Lemma 36. *(i) d is linear on $\Lambda^k M$, that is, $d(\lambda\omega + \eta) = \lambda d\omega + d\eta$ for $\omega, \eta \in \Lambda^k M$, $\lambda \in \mathbb{R}$, and $(X_1 \dots, X_k) \mapsto d\omega(X_1, \dots, X_k)$ is $C^\infty(M)$-multilinear.*
(ii) A product rule holds:

$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta \qquad \forall \omega \in \Lambda^k M,\ \eta \in \Lambda^l M. \tag{48}$$

(iii) $d^2 = 0$, i.e., $d(d\omega) = 0$ $\forall \omega \in \Lambda^k M$.

Specifically for $f \in C^\infty(M)$ the product rule gives $d(f\omega) = df\omega + fd\omega$.

Proof. (i) $\mathbb{R}$-linearity follows from $\omega_I \mapsto \frac{\partial \omega_I}{\partial x^r}$ being $\mathbb{R}$-linear. The $C^\infty(M)$-linearity in each entry X_i holds for all basis elements $e^r \wedge e^I$.
(ii) By $\mathbb{R}$-linearity it suffices to check this for $\omega = ge^I$ and $\eta = he^J$:

$$\begin{aligned} d(\omega \wedge \eta) = d(ge^I \wedge he^J) &= d(gh) \wedge e^I \wedge e^J = \big((dg)h + g\, dh\big) \wedge e^I \wedge e^J \\ &= (dg \wedge e^I) \wedge he^J + (-1)^k ge^I \wedge (dh \wedge e^J). \end{aligned}$$

(iii) Again by $\mathbb{R}$-linearity it suffices to consider $\omega = fe^I$ in which case

$$d\omega = \sum_{r=1}^n \frac{\partial f}{\partial x^r} e^r \wedge e^I \quad \Rightarrow \quad d(d\omega) = \sum_{r=1}^n \sum_{s=1}^n \frac{\partial^2 f}{\partial x^s \partial x^r} e^s \wedge e^r \wedge e^I.$$

Due to $e^s \wedge e^r = -e^r \wedge e^s$ and the Schwarz Theorem $\partial_{sr} f = \partial_{rs} f$, the terms of the sum cancel in pairs. $\square$

In particular, $d^2 f = 0$ is equivalent to the Schwarz Theorem. Note that writing d^2 is somewhat inaccurate, since the two differentials involved are maps between different spaces.

Integrability conditions can often be stated most elegantly in terms of forms. For instance, the Frobenius theorem can be formulated as follows: A distribution Δ^k is integrable if and only if the ideal

$$\mathcal{I}(\Delta) := \{\omega \in \Lambda^l(M) : \omega(X_1, \ldots, X_l) = 0 \text{ if } X_1(p), \ldots, X_l(p) \in \Delta^k(p) \ \ \forall p \in M\}$$

satisfies $d\big(\mathcal{I}(\Delta)\big) \subset \mathcal{I}(\Delta)$. It is a good exercise to check this statement on examples of distributions.

To see that d is well-defined on a manifold, we state an invariant formula for d. To denote omission of an entry we use a hat $\widehat{\cdot}$.

Proposition 37. *For each k-form ω and $k+1$ vector fields X_i, the right hand side of*

$$\begin{aligned} d\omega(X_1, \ldots, X_{k+1}) = &\sum_{i=1}^{k+1} (-1)^{i+1} \partial_{X_i}\big(\omega(X_1, \ldots, \widehat{X_i}, \ldots, X_{k+1})\big) \\ &+ \sum_{1 \le i < j \le k+1} (-1)^{i+j} \omega\big([X_i, X_j], X_1, \ldots, \widehat{X_i}, \ldots, \widehat{X_j}, \ldots, X_{k+1}\big) \end{aligned} \tag{49}$$

has the local representation (47)*. Therefore,* (47) *is well-defined globally.*

Let us exhibit the most frequently applied form of (49), which is the case $k = 1$. Then the first sum of (49) contains the two terms $i = 1, 2$ and the second sum only one ($i = 1, j = 2$):

$$d\omega(X, Y) = \partial_X\big(\omega(Y)\big) - \partial_Y\big(\omega(X)\big) - \omega\big([X, Y]\big) \tag{50}$$

Remark. If all X_i in (49) are coordinate vector fields, then $d\omega$ is given by the first sum alone. Thus we can regard the second sum as a correction term necessary for the invariance of d when we pass from coordinate to arbitrary vector fields. This insight is the key idea for the following proof. Let us also note that in this way the Lie bracket appears in other invariant formulas: a famous example is the Riemann curvature tensor of a Riemannian manifold.

Proof. First we claim that the right hand side of (49) is $C^\infty(M)$-multilinear in the X_i. We prove this only for (50). Then $C^\infty(M)$-multilinearity in X amounts to the vanishing of

$$d\omega(fX, Y) - f d\omega(X, Y) = 0 - \partial_Y(f\omega(X)) + f\partial_Y(\omega(X)) - \omega\big([fX, Y] - f[X, Y]\big).$$

Indeed, due to $[fX, Y] = f[X, Y] - (\partial_Y f)X$ this does vanish. By anti-commutativity of the right hand side of (50) the same holds w.r.t. Y. The proof of the general case $k \ge 1$ is essentially the same but involves more indices.

We must prove that locally, w.r.t. any chart (x, U), the right hand side of (49) agrees with (47). We simplify our task:

• Using additivity of (47) and (49) in ω, it is sufficient to consider the case $\omega = fe^I$. By coordinate renumbering we may assume specifically $\omega = fe^I = fe^1 \wedge \ldots \wedge e^k$.

• Also, (47) and (49) are $C^\infty(M)$-multilinear in the X_i's. So it suffices to check the identity for the case of standard basis multivectors $(X_1, \ldots, X_{k+1}) = (e_{j_1}, \ldots, e_{j_{k+1}})$. Since the forms are alternating we can assume that the indices are increasing, $j_1 < \ldots < j_{k+1}$.

We evaluate (49) for this case. For standard basis vectors the associated flows commute and we have $[e_i, e_j] = 0$ by Thm. 25. Therefore the second sum of (49) vanishes and so

$$d\omega(e_{j_1}, \ldots, e_{j_{k+1}}) = \sum_{i=1}^{k+1} (-1)^{i+1} \partial_{e_{j_i}} \big(fe^1 \wedge \cdots \wedge e^k \, (e_{j_1}, \ldots, e_{j_{i-1}}, \widehat{e_{j_i}}, e_{j_{i+1}}, \ldots, e_{j_{k+1}}) \big). \tag{51}$$

But $\omega = fe^1 \wedge \cdots \wedge e^k$ vanishes identically on all ordered multivectors except for $(e_1, \ldots, e_k)$. Therefore, nonzero contributions only arise with the cancelled vector $\widehat{e_{j_i}}$ occuring in the last position. That is, only $i = k+1$ contributes to the sum. Therefore, only those terms in (51) are nonzero whose indices satisfy

$$(j_1, \ldots, j_k, j_{k+1}) \in \big\{ (1, \ldots, k, r) : k+1 \le r \le n \big\}. \tag{52}$$

That is, the only nonzero terms in (51) are

$$d\omega(e_1, \ldots, e_k, e_r) = (-1)^{k+2} \partial_{e_r} \big(\omega(e_1, \ldots, e_k) \big) = (-1)^k \frac{\partial f}{\partial x^r}, \quad r = k+1, \ldots, n. \tag{53}$$

On the other hand, (47) gives for our $\omega = fe^I$ that

$$d\omega = df \wedge e^I = \sum_r \frac{\partial f}{\partial x^r} \, e^r \wedge e^I = (-1)^k \sum_r \frac{\partial f}{\partial x^r} \, e^I \wedge e^r.$$

The terms with $r = 1, \ldots, k$ vanish, and so the contributing indices (I, r) are again given by (52). Hence evaluated on $(e_1, \ldots, e_k, e_r)$ this agrees with (53). $\square$

There is one more important property of d:

Theorem 38. *If $f \colon M \to N$ is differentiable and $\omega \in \Lambda^k N$ then*

$$f^*(d\omega) = d(f^*\omega). \tag{54}$$

As Palais proved in 1959, the only operator from $\Lambda^k M$ to $\Lambda^{k+1} M$ which commutes with f^* as in (54) is d, up to a constant multiple (see [Sp I, p.307]).

Proof. Recall that f^* acts as follows: First, the footpoint of the form p is replaced by $f(p)$. Second, in the multi-vector argument, each X_i gets replaced by $df X_i$.

We use induction on k. For $k=0$, we have for $\omega = g \in C^\infty(M)$, as desired,

$$f^*(dg)(X) \overset{\text{def.} f^*}{=} dg\big(df(X)\big) = d(g\circ f)(X) = d(f^*g)(X).$$

To verify the last equality sign note that for a 0-form, f^* only replaces the footpoint.

Assuming the formula for $k-1$, it is sufficient to consider $\omega = g e^{i_1} \wedge \cdots \wedge e^{i_k}$: Note first that $de^{i_k} = 0$ and so $d(f^* e^{i_k}) = de^{i_k} \circ df = 0$. This gives the term "+0" after the second equality sign in

$$\begin{aligned} d(f^*\omega) &= d\big(f^*(g\, e^{i_1} \wedge \cdots \wedge e^{i_{k-1}}) \wedge f^* e^{i_k}\big) \overset{(48)}{=} d\big(f^* g\, e^{i_1} \wedge \cdots \wedge e^{i_{k-1}}\big) \wedge f^* e^{i_k} + 0 \\ &\overset{\text{ind. hypoth.}}{=} f^*\big(d(g\, e^{i_1} \wedge \cdots \wedge e^{i_{k-1}})\big) \wedge f^* e^{i_k} \;=\; f^*\big(dg \wedge e^{i_1} \wedge \cdots \wedge e^{i_{k-1}}\big) \wedge f^* e^{i_k} \\ &= f^*(d\omega) \end{aligned}$$

□

Example. Let $f\colon \mathbb{R}^2 \to \mathbb{R}^2$, $f(u,v) = (u^2, uv)$ and $\omega \in \Lambda^1\mathbb{R}^2$ with $\omega_{(x,y)} = \cos y\, dx + x\, dy$. We can use the theorem to compute the pullback of the form:

$$\begin{aligned} f^*\omega &= ((\cos y)\circ f) f^* dx + (x\circ f) f^* dy \\ &\overset{(54)}{=} \cos(uv)\, d(f^*x) + u^2\, d(f^*y) = \cos(uv)\, d(x\circ f) + u^2\, d(y\circ f) \\ &= \cos(uv)\, d(u^2) + u^2 d(uv) = (2u\cos u + u^2 v)\, du + u^3\, dv \end{aligned}$$

Remark. We have seen that a form $\eta := d\omega$ must satisfy $d\eta = 0$. Let us introduce common terminology:
- A form η with $d\eta = 0$ is called *closed*,
- a form η, such that ω with $\eta = d\omega$ exists, is called *exact*.

Our initial statement can be rephrased to say that an exact form is closed.

Conversely, one might ask if a closed form is exact, that is, if $d\eta = 0$ implies the existence of ω with $d\omega = \eta$. On a domain or manifold which is *contractible* the *Poincaré Lemma* assures that a closed form is indeed exact. Without any requirement on the domain this cannot be true, as the following example indicates. On a slit domain in $\mathbb{R}^2$ consider the polar angle function φ. Then $\eta =: d\varphi$ is a 1-form such that $d\eta = d^2\varphi$ vanishes; this is also true (by continuity) on the domain $\mathbb{R}^2 \setminus \{0\}$. Nevertheless, φ itself is no longer defined on this domain. So η is a 1-form which is closed but not exact on $\mathbb{R}^2 \setminus \{0\}$.

7. Integration of differential forms over chains

7.1. **Integration over cubes.** For a motivation, let $U \subset \mathbb{R}^n$ and consider the path integral of a vector field $X \in \mathcal{V}(U)$ along a curve $c\colon I \to U$,

$$\int_c X\, ds := \int_I \langle X(c(t)), c'(t)\rangle\, dt. \tag{55}$$

The chain rule gives the integral is parameterization invariant; indeed, for $\varphi\colon \tilde I \to I$,

$$\int_{\tilde I} \big\langle X(c(\varphi(\tau))), (c\circ\varphi)'(\tau)\big\rangle\, d\tau = \int_{\tilde I} \big\langle X(c(\varphi(\tau))), c'(\varphi(\tau))\big\rangle \varphi'(\tau)\, d\tau = \int_I \big\langle X(c(t)), c'(t)\big\rangle\, dt.$$

In generalization of (55) we can define the integral of a 1-form $\omega\colon U \times \mathbb{R}^n \to \mathbb{R}$ by

$$\int_c \omega := \int_I \omega_{c(t)}(c'(t))\, dt = \int_I c^*\omega. \tag{56}$$

This includes (55), setting $\omega_p(\cdot) := \langle X(p), \cdot\rangle$. Using the $C^\infty(M)$-linearity of $Y \mapsto \omega_p(Y)$, the above calculation proves also the parameterization invariance of the form integral (56).

We now want to integrate k-forms over k-dimensional submanifolds. While we have described manifolds in terms of charts, our one-dimensional example indicates that for the present task it is more natural to consider parameterizations. We consider local parameterizations first. Since integration wll be seen to be parameterization independent, we can later piece such parameterizations together using a partition of unity.

Definition. A differentiable map $\sigma\colon [0,1]^k \to M^n$, $k \in \mathbb{N}$, is called a *(singular) k-cube* in M. For $k = 0$ we write $[0,1]^0 = \{0\}$, and so $\sigma\colon \{0\} \to M$.

Note that our standing assumption for differentiability on non-open sets is that there exists a differentiable extension to some open superset.

Example. A 1-cube is a differentiable curve.

We generalize (56) to the case of k-forms, first for $\mathbb{R}^k$ and then for k-cubes in manifolds:

Definition. (i) If $K \subset \mathbb{R}^k$ is compact and $\eta(x) = f(x)\, e^1 \wedge \cdots \wedge e^k \in \Lambda^k K$ then

$$\int_K \eta := \int_K f(x)\, dx.$$

(ii) For ω a k-form on a manifold M^n and σ a k-cube in M let

$$\int_\sigma \omega := \int_{[0,1]^k} \sigma^*\omega,$$

where the right hand side is defined by (i). For $k = 0$ we set $\int_\sigma f = f(\sigma(0))$.

Example. Consider the hyperbolic paraboloid $\sigma\colon [0,1]^2 \to \mathbb{R}^3$, $\sigma(x,y) = (x, y, xy)$. If we denote the $\mathbb{R}^2$-basis by $b_1 = (1,0)$ and $b_2 = (0,1)$ then $d\sigma_{(x,y)} b_1 = (1,0,y)$ and $d\sigma_{(x,y)} b_2 = (0,1,x)$. Now consider the 2-form $e^1 \wedge e^2 \in \Lambda^2\mathbb{R}^3$, where (e^1, e^2, e^3) is the standard dual basis. This form measures the area of the (x,y)-projection: indeed its pullback satisfies

$$\big(\sigma^*(e^1 \wedge e^2)\big)(b_1, b_2) = (e^1 \wedge e^2)(d\sigma\, b_1, d\sigma\, b_2) = (e^1 \wedge e^2)\big((1,0,y),(0,1,x)\big) \overset{(43)}{=} 1,$$

that is, $\sigma^*(e^1 \wedge e^2) = b^1 \wedge b^2$. Thus $\int_{[0,1]^2} \sigma^*(e^1 \wedge e^2) = \int_{[0,1]^2} 1\, dxdy = 1$ is the area of a unit square. On the other hand, for $e^2 \wedge e^3$ we find $(\sigma^*(e^2 \wedge e^3))(b_1, b_2) = -y$ and so

$\int_{[0,1]^2} \sigma^*(e^2 \wedge e^3) = -\int_{[0,1]^2} y\,dxdy = -1/2$, which again agrees with the signed area of the relevant projection, here a triangle.

Remarks. 1. Writing $dx^i := e^i$ on $\mathbb{R}^n$, the two sides of (i) appear almost alike:

$$\int_K f\,dx^1 \wedge \cdots \wedge dx^n = \int_K f\,dx^1 \dots dx^n.$$

However, for $\eta = f\,dx^2 \wedge dx^1 \in \Lambda^2\mathbb{R}^2$ we obtain $\int \eta = \int -f\,dx^2 dx^1$, meaning that the form integral in (i) is sensitive to orientation. Thus if in (ii) we replace σ by the orientation reversing cube $\tilde{\sigma}(x_1, \dots, x_n) := \sigma(x_2, x_1, x_3, \dots, x_n)$ then $\int_{\tilde{\sigma}} \omega = -\int_\sigma \omega$.
2. $\int_\sigma \omega$ counts the image with multiplicity: If σ covers a set twice with the same orientation then each image point contributes twice to the integral; if σ covers twice with opposite orientation the integral vanishes.
3. The cube σ need not be an immersion. For instance, it is admissible that $\sigma([0,1]^k)$ is contained in a $(k-1)$-dimensional submanifold, in which case the integral vanishes.

Under a change of charts, a k-form transform with the determinant of the Jacobian of the transition map, see Proposition 34. On the other hand, the change of variables formula for integration has precisely the same dependence. Therefore, the integral of a form is parameterization independent, provided the change of parameter change is orientation preserving:

Lemma 39. *If $\tau\colon [0,1]^k \to [0,1]^k$ is a diffeomorphism with $\det d\tau > 0$ and $\eta \in \Lambda^k([0,1]^k)$ then $\int_{[0,1]^k} \tau^*\eta = \int_{[0,1]^k} \eta$.*

Proof. Writing $\eta(x) = f(x)\,e^K$ where $e^K = e^1 \wedge \cdots \wedge e^k$ we obtain

$$\int_{[0,1]^k} \tau^*(fe^K) \overset{\text{Thm. 34}}{=} \int_{[0,1]^k} (f \circ \tau) \det d\tau\, e^K = \int_{[0,1]^k} (f \circ \tau) |\det d\tau|\, dx$$
$$\overset{\text{ch. of var.}}{=} \int_{\tau([0,1]^k)} f(x)\,dx = \int_{\tau([0,1]^k)} fe^K. \qquad \square$$

Proposition 40. (i) *For σ a k-cube, τ as in the lemma, and $\omega \in \Lambda^k M$ we have*

$$\int_\sigma \omega = \int_{\sigma\circ\tau} \omega.$$

(ii) *If σ_1, σ_2 are singular k-cubes, such that $\tau := \sigma_2^{-1} \circ \sigma_1\colon A := \sigma_1^{-1}(\sigma_2([0,1]^k)) \to [0,1]^k$ satisfies $\det d\tau > 0$. Then, for $\omega \in \Lambda^k M$,*

$$\operatorname{supp}\omega \subset \sigma_1([0,1]^k) \cap \sigma_2([0,1]^k) \qquad \textit{implies} \qquad \int_{\sigma_1} \omega = \int_{\sigma_2} \omega.$$

Proof. (i) Applying the lemma to $\eta := \sigma^*\omega$ we find

$$\int_{\sigma\circ\tau} \omega = \int_{[0,1]^k} (\sigma\circ\tau)^*\omega = \int_{[0,1]^k} \tau^*(\sigma^*\omega) \overset{\text{Lemma}}{=} \int_{\tau([0,1]^k)=[0,1]^k} \sigma^*\omega = \int_\sigma \omega$$

(ii) Apply the first part to $\sigma := \sigma_2$ and τ, by restricting it to the set A. This gives

$$\int_A \sigma_1^*\omega = \int_A (\sigma\circ\tau)^*\omega \overset{(i)}{=} \int_{\tau(A)} \sigma^*\omega. = \int_{\tau(A)} \sigma_2^*\omega.$$

The assumption on the support of ω gives that the left hand side agrees with $\int_{[0,1]^k} \sigma_1^*\omega = \int_{\sigma_1} \omega$, and the right hand side with $\int_{[0,1]^k} \sigma_2^*\omega = \int_{\sigma_2} \omega$. □

7.2. **Chains.** Stokes' theorem involves integration over the boundary of a k-cube. A k-cube has $2k$ bounding faces, where each face is a $(k-1)$-cube. In order to integrate over the boundary we will simply add up the $2k$ integrals over the faces. It is useful to do this in more generality. Not only will we integrate forms along unions of cubes by representing the union formally a sum, but we will take the algebraic closure of these sums, defining thereby a vector space:

Definition. (i) A *k-chain* σ in M^n is a formal linear combination of k-cubes, $\sigma = \sum_{i=1}^l a^i\sigma_i$, where $a^i \in \mathbb{R}$.
(ii) The integral of a k-form ω over the k-chain σ is given by

$$\int_{\sum a^i\sigma_i} \omega := \sum_{i=1}^l \Big(a^i \int_{\sigma_i} \omega\Big). \tag{57}$$

Note that $\sum a_i\sigma_i$ does not assign a value to $(x_1, \dots, x_k)$, so that σ is not a map. Instead, it is a purely formal notion in order to introduce (57).

Example. We can now write $\int_{-\sigma} \omega = -\int_\sigma \omega$ where integration over $-\sigma$ has the same effect as changing the orientation of σ.

We now associate to a k-cube its chain of boundary $(k-1)$-cubes. We denote the $2k$ faces which result as a boundary restriction from σ in terms of double indices (j,b): The two parallel faces with normal e_j are distinguished by b. That is, given $\sigma\colon [0,1]^k \to M$ we define for $1 \le j \le k$ and $b \in \{0,1\}$

$$\sigma_{j,b}\colon [0,1]^{k-1} \to M, \qquad \sigma_{j,b}(x^1, \dots, x^{k-1}) := \sigma\big(x^1, \dots, x^{j-1}, b, x^j, \dots x^{k-1}\big). \tag{58}$$

Examples. 1. In case of a 3-cube, $\sigma_{3,0}$ parameterizes with the bottom face $\{z=0\}$, while $\sigma_{3,1}$ parameterizes with the top face $\{z=1\}$.
2. The endpoints of a 1-cube or curve σ are $\sigma_{1,0} = \sigma(0)$ and $\sigma_{1,1} = \sigma(1)$.

We need to take orientation into account. The intuitive notion of arrows on edges or of rotation senses on faces is made rigorous by saying that either the orientation is consistent with the parameterization $\sigma_{j,b}$ or not. Our convention is such that each pair of opposite parallel faces $\sigma_{j,0}$ and $\sigma_{j,1}$ will be assigned opposite orientations:

Definition. (i) The *boundary of a k-cube* σ in M^n with $k \in \mathbb{N}$ is the $(k-1)$-chain

$$\partial\sigma := \sum_{j=1}^{k} \sum_{b\in\{0,1\}} (-1)^{j+b}\sigma_{j,b}.$$

For $k = 0$, when $\sigma\colon [0,1]^0 = \{0\} \to M$, we define $\partial\sigma := 0 \in \mathbb{R}$.
(ii) The *boundary of a k-chain* $\sigma = \sum_{i=1}^{l} a^i\sigma_i$, where $k \in \mathbb{N}_0$, is

$$\partial\sigma := \sum_{i=1}^{l} a^i \partial\sigma_i.$$

(iii) A chain σ is *closed* if $\partial\sigma = 0$.

Examples. 1. A closed 1-cube σ is a closed curve. Indeed, $\partial\sigma = \sigma_{1,1} - \sigma_{1,0} = \sigma(1) - \sigma(0)$ vanishes if and only if $\sigma(0) = \sigma(1)$.
2. Two curves σ^1, σ^2 with the same endpoints form a 1-chain $\sigma := \sigma^1 - \sigma^2$ which is also closed. Indeed, if $\sigma^1(0) = \sigma^2(0)$ and $\sigma^1(1) = \sigma^2(1)$ then

$$\partial\sigma = \sigma^1_{1,1} - \sigma^1_{1,0} - \sigma^2_{1,1} + \sigma^2_{1,0} = 0.$$

3. Let $\sigma\colon [0,1]^2 \to M$ be a 2-cube. We claim that $\partial\sigma$ is closed. To see this, consider the edges from the origin in counterclockwise order, $\partial\sigma = \sigma_{2,0} + \sigma_{1,1} - \sigma_{2,1} - \sigma_{1,0}$, and label the vertices as $P = \sigma(0,0)$, $Q = \sigma(1,0)$ etc. Then indeed

$$\partial(\partial\sigma) = (Q-P) + (R-Q) - (R-S) - (S-P) = 0.$$

4. For a 3-cube the 1-chain $\partial(\partial\sigma)$ is a sum over $24 = 6\cdot 4$ edges of the cube, where it can be checked that each edge appears with two opposite signs. So again $\partial(\partial\sigma) = 0$.

The geometric property $\partial(\partial\sigma) = 0$, verified in the above examples, is analogous to $d^2 = 0$ for forms, and holds in general:

Proposition 41. *If σ is a k-chain in M^n, then*

$$\partial^2\sigma := \partial(\partial\sigma) = 0.$$

Proof. It is sufficient to check this for a k-cube σ whose double boundary is

$$\partial(\partial\sigma) = \partial\Big(\sum_{i=1}^{k} \sum_{a\in\{0,1\}} (-1)^{i+a}\sigma_{i,a}\Big) = \sum_{j=1}^{k-1} \sum_{b\in\{0,1\}} \sum_{i=1}^{k} \sum_{a\in\{0,1\}} (-1)^{i+j+a+b}(\sigma_{i,a})_{j,b}. \tag{59}$$

We want to verify that the sum contains pairs of equal terms with different signs.

Note that $\sigma_{i,\alpha}$ parameterizes the $2k$ facets of σ, which are $(k-1)$-cubes, and so $(\sigma_{i,a})_{j,b}(x)$ restricts the parameterization of $\sigma_{i,\alpha}$ to the bounding $(k-2)$-cubes. Let us consider these $(k-2)$-cubes first for the subset of indices $I_1 := \{(i,j) : 1 \le i \le j \le k-1\}$. Then, for $x \in [0,1]^{k-2}$,

$$\begin{aligned}(\sigma_{i,a})_{j,b}(x) &= \sigma_{i,a}(x^1, \dots, x^{j-1}, b, x^j, \dots, x^{k-2}) \\ &= \sigma(x^1, \dots, x^{i-1}, a, x^i, \dots, x^{j-1}, b, x^j, \dots, x^{k-2}) \\ &= \sigma_{j+1,b}(x^1, \dots, x^{i-1}, a, x^i, \dots, x^{k-2}) \;=\; (\sigma_{j+1,b})_{i,a}(x)\end{aligned}$$

Since the set of indices (i,j) appearing in the last expression satisfies $1 \le i < j+1 \le k$, these terms appear in the sum (59) as labelled by $(\sigma_{i,a})_{j,b}$ where (i,j) are in the set $I_2 := \{(i,j) : 1 \le j < i \le k\}$, and a is replaced by b.

But the entire index set $I := \{(i,j) : 1 \le i \le k \text{ and } 1 \le j \le k-1\}$ can be written as a disjoint union $I = I_1 \cup I_2$, and the index change of the last paragraph, that is, $(i,j) \mapsto (j+1,i)$, gives a bijection between I_1 and I_2. Finally, under the bijection $(i,j,a,b) \mapsto (j+1,i,b,a)$ the sign in (59) changes, so that the sum consists of pairs of cancelling terms. $\square$

Remarks. 1. The essential feature in the definition of ∂ is the sign. The proposition tells us that the exponent $j+b$ is chosen in a way as to endow each pair of $(k-2)$-dimensional faces of a k-cube with the opposite orientation.

2. While k-cubes are well adapted to coordinate parallel integration as needed for Stokes' Theorem, there is another setting more widely used in algebraic topology: *k-simplices* are build from triangles rather than squares, and have a similar boundary operator ∂. See [W], for instance, for this approach.

3. More generally, a family of groups G_k where $k \in \mathbb{N}_0$ and a group homomorphism $d\colon G_k \to G_{k+1}$ with $d^2 = 0$ is called a *chain complex*; an example is $G_k = \Lambda^k M$ with addition. Similarly, in case $d\colon G_k \to G_{k-1}$ the groups are called a *cochain complex*; an example for G_k are k-chains. On (co-)chain complexes, a homology theory can be defined.

7.3. **Stokes' Theorem for chains.** We prove a parameterized version of Stokes' theorem:

Theorem 42. *If σ is a k-chain in a manifold M^n and ω is a $(k-1)$-form on M then*

$$\int_\sigma d\omega = \int_{\partial\sigma} \omega.$$

Proof. 1. Consider the case $M = \mathbb{R}^n$, $k = n$, the standard n-cube $\mathrm{id}^n := \mathrm{id}|_{[0,1]^n}$ in $\mathbb{R}^n$. By linearity it is sufficient to prove the theorem for the particular $(n-1)$-forms

$$\omega = f e^1 \wedge \cdots \wedge \widehat{e^i} \wedge \cdots \wedge e^n, \qquad \text{where } f\colon [0,1]^n \to \mathbb{R},\ \ 1 \le i \le n.$$

Let us first compute the left hand side. Since

$$d\omega = \partial_i f\, e^i \wedge e^1 \wedge \cdots \wedge \widehat{e^i} \wedge \cdots \wedge e^n = (-1)^{i-1} \partial_i f\, e^1 \wedge \cdots \wedge e^n,$$

We apply the fundamental theorem of calculus to integration w.r.t. the i-th variable, after invoking Fubini's theorem. This gives

$$\begin{aligned} \int_{[0,1]^n} d\omega &= \int_{[0,1]^n} (-1)^{i-1} \partial_i f(x^1, \dots, x^i, \dots, x^n)\, dx^1 \dots dx^i \dots dx^n \\ &= (-1)^{i-1} \int_{[0,1]^{n-1}} f(x^1, \dots, 1, \dots, x^n) - f(x^1, \dots, 0, \dots, x^n)\, dx^1 \dots \widehat{dx^i} \dots dx^n \\ &= (-1)^{i-1} \int_{[0,1]^{n-1}} f(y^1, \dots, 1, \dots, y^{n-1}) - f(y^1, \dots, 0, \dots, y^{n-1})\, dy^1 \dots dy^{n-1}. \end{aligned} \tag{60}$$

Now we compute the right hand side. As in (58), we have that our cube $\sigma = \mathrm{id}^n$ has its j-th pair of bounding faces parameterized by

$$\sigma_{j,b} \colon [0,1]^{n-1} \to [0,1]^n, \qquad \sigma_{j,b}(x^1, \dots, x^{n-1}) := \big(x^1, \dots, x^{j-1}, b, x^j, \dots x^{n-1}\big). \tag{61}$$

Then $\partial\, \mathrm{id}^n = \sum_{j,b} (-1)^{j+b} \sigma_{j,b}$ and so, by definition of the chain integral,

$$\int_{\partial\, \mathrm{id}^n} \omega = \sum_{j=1}^{n} \sum_{b \in \{0,1\}} (-1)^{j+b} \int_{\sigma_{j,b}} \omega = \sum_{j=1}^{n} \sum_{b \in \{0,1\}} (-1)^{j+b} \int_{[0,1]^{n-1}} (\sigma_{j,b})^* f e^1 \wedge \cdots \wedge \widehat{e^i} \wedge \cdots \wedge e^n.$$

We claim that all terms with $j \neq i$ vanish,

$$(\sigma_{j,b})^* \big(e^1 \wedge \cdots \wedge \widehat{e^i} \wedge \cdots \wedge e^n\big) = 0 \qquad \text{for all } j \neq i.$$

Geometrically this is the fact that ω vanishes on all multivectors tangent to all cube faces except for the i-th one. To verify this fact by calculation, take partial derivatives of (61):

$$d\sigma_{j,b}(e_1) = e_1,\ \dots,\ d\sigma_{j,b}(e_{j-1}) = e_{j-1}, \quad d\sigma_{j,b}(e_j) = e_{j+1},\ \dots,\ d\sigma_{j,b}(e_{n-1}) = e_n \tag{62}$$

Since e_j is not in the image, $e^j(d\sigma_{j,b}(e_k)) = 0$ for all $k = 0, \dots, n-1$, implying our claim.

It remains to consider $j := i$. In view of (62) we have

$$e^1 \circ d\sigma_{i,b} = e^1, \ \dots, \ e^{i-1} \circ d\sigma_{i,b} = e^{i-1}, \quad e^{i+1} \circ d\sigma_{i,b} = e^i, \ \dots, \ e^n \circ d\sigma_{i,b} = e^{n-1}.$$

To verify these equalities assert that both sides act the same way on the basis vectors $(e_1, \dots, e_{n-1})$. We conclude

$$\int_{\partial\, \mathrm{id}^n} \omega = \sum_{b \in \{0,1\}} (-1)^{i+b} \int_{[0,1]^{n-1}} f\big(x^1, \dots, x^{i-1}, b, x^i, \dots x^{n-1}\big)\, e^1 \wedge \cdots \wedge e^{n-1}.$$

Our result agrees with (60) and we have established Stokes' theorem for our special case.

2. For σ a k-cube in M^n we can apply Step 1 to the k-form $\sigma^*\omega$ on $\mathbb{R}^k$

$$\int_\sigma d\omega \overset{\text{def.}}{=} \int_{[0,1]^k} \sigma^*(d\omega) \overset{\text{Thm.38}}{=} \int_{[0,1]^k} d(\sigma^*\omega) \overset{1.}{=} \int_{\partial\, \mathrm{id}^k} \sigma^*\omega \overset{\text{def.}}{=} \int_{\partial\sigma} \omega,$$

where in fact at the last equality the definition of the form integral is applied to a chain.

3. The generalization to σ a k-chain is immediate. $\square$

Note that the manifold dimension n can be larger than k, but the integral will not see the extra dimensions.

Examples. 1. In case $k = 1$ we can apply the theorem to a 0-form f, and the 1-cube id^1. Then $\int_{\mathrm{id}^1} df = \int_{\partial\,\mathrm{id}^1} f$ which means $\int_{[0,1]} f'(x)\,dx = f(1) - f(0)$.

2. The *Divergence Theorem* follows from taking an alternating sum over the forms used in Step 1. Specifically, consider for $f = (f_1, \dots, f_n)\colon [0,1]^n \to \mathbb{R}^n$ the $(n-1)$-form

$$\omega = \sum_i (-1)^{i+1} f_i\, e^1 \wedge \cdots \wedge \widehat{e^i} \wedge \cdots \wedge e^n \quad \text{with} \quad d\omega = \operatorname{div} f\, e^1 \wedge \cdots \wedge e^n.$$

Writing $\omega = \sum_i \omega_i$ note that precisely ω_i was considered in the previous proof. Therefore, only the two i-faces $\sigma_{i,b}$ of $[0,1]^n$ contribute to $\int_{\partial[0,1]^n} \omega_i$. Hence the boundary integral reads

$$\begin{aligned}\int_{\partial\,\mathrm{id}^n} \omega &= \sum_{i,b} (-1)^{i+b} \int_{\sigma_{i,b}} \omega_i = \sum_{i,b} (-1)^{i+b} \int_{\sigma_{i,b}} (-1)^{i+1} f_i\, e^1 \wedge \cdots \wedge \widehat{e^i} \wedge \cdots \wedge e^n \\ &= \sum_{i,b} (-1)^{b+1} \int_{[0,1]^{n-1}} f_i\big(\sigma_{i,b}(x)\big)\, e^1 \wedge \cdots \wedge e^{n-1} = \sum_{i,b} \int_{[0,1]^{n-1}} \langle f \circ \sigma_{i,b},\ \nu_i \rangle\, dx,\end{aligned}$$

where ν_i is the exterior normal to the face $\sigma_{i,b}$, that is, $\nu_i = e_i$ on $\sigma_{i,1}$ and $\nu_i = -e_i$ on $\sigma_{i,0}$. We may write ν for the exterior normal on all faces, where ν is defined except on the $(k-2)$-dimensional faces of the cube which have measure 0. Then we the right hand side becomes a surface integral, namely $\int_{\partial[0,1]^n} \langle f, \nu \rangle\, dS$, noting that $\sigma_{i,b}$ is a parameterization with Gram determinant 1. We conclude that for the case considered, Stokes' theorem gives the classical divergence theorem,

$$\int_{[0,1]^n} \operatorname{div} f\, dx = \int_{\partial[0,1]^n} \langle f, \nu \rangle\, dS.$$

We leave it to the reader to extend this formula to the case of an n-cube $\sigma\colon [0,1]^n \to \mathbb{R}^n$ which is a diffeomorphism onto its image.

8. Integration of forms over manifolds with boundary

8.1. **Integration over manifolds.** The integral $\int_\sigma \omega$ is sensitive to orientation, meaning that it changes sign when we replace σ by $\tilde\sigma(x_1, \dots, x_k) = \sigma(1 - x_1, x_2, \dots, x_k)$. For this reason we need to deal with orientation issues if we are to define the integral of a form over a manifold.

Definition. (i) Two charts (x, U) and (y, V) of a manifold M are *orientation compatible* if the transition map satisfies

$$\det d(y \circ x^{-1}) > 0 \quad \text{for all } p \in x(U \cap V). \tag{63}$$

(ii) An *orientation* of M is an atlas $\mathcal{A} = \{(x_\alpha, U_\alpha) : \alpha \in A\}$ whose charts are pairwise orientation compatible.
(iii) M is *orientable* if it has an orientation.

Example. Möbius band and Klein bottle are non-orientable 2-manifolds, $\mathbb{R}P^n$ is a non-orientable n-manifold.

Note that orientation compatibility is an equivalence relation, in particular $\det d(y \circ x^{-1}) > 0 \Leftrightarrow \det d(x \circ y^{-1}) > 0$ by the Inverse Function Theorem.

The following is not hard to check:
• If two charts with a connected nonempty intersection set are not orientation compatible then composing one chart with an orientation reversing diffeomorphism of $\mathbb{R}^n$ makes them orientation compatible.

• Suppose M is connected and $(M, \mathcal{A})$ is orientable. If an additional chart has a nonempty intersection set with a particular chart of $\mathcal{A}$ and is orientation compatible with it, then the additional chart will be orientation compatible with all charts of $\mathcal{A}$. To prove this, use that the left hand side of (63) is continuous by our general assumptions.

• On a connected orientable manifold, there are exactly two differentiable structures $(M, \mathcal{S}_+)$, $(M, \mathcal{S}_-)$ which give an orientation. More generally, an orientation can be chosen for each connected component.

On an oriented manifold $(M, \mathcal{A})$, we call an n-tuple of linearly independent tangent vectors $(v_1, \ldots, v_n) \in (T_pM)^n$ *(positively) oriented* if the orientation of their principal parts,

$$\mu(v_1, \ldots, v_n) := \operatorname{sign}\big(\det(dx(v_1), \ldots, dx(v_n))\big) \tag{64}$$

is $+1$ for $x \in \mathcal{A}$.

Given an oriented manifold $(M^n, \mathcal{A})$, we call a local diffeomorphism $\tau\colon \Omega \subset \mathbb{R}^n \to M$ *orientation preserving* if τ^{-1} is orientation compatible with $(M, \mathcal{A})$, and *orientation reversing* if it is not. It is the statement of Proposition 40(ii) that $\int_M \omega := \int_\sigma \omega$ is well-defined for all diffeomorphic n-cubes σ, compatible with the orientation of M, such that $\operatorname{supp} \omega \subset \sigma([0,1]^n)$.

Consider an oriented manifold $(M^n, \mathcal{A})$. By modifying charts suitably, one can show there is a countable open covering $\{U_\alpha : \alpha \in \mathcal{A}\}$ of M with charts (x_α, U_α) such that each U_α is contained in the image of an orientation preserving diffeomorphic n-cube. Indeed, we could cover $x_\alpha(U)$ with coordinate parallel open cubes in a locally finite way. The restriction of the x_α to these cubes then defines charts which cover, i.e. define an atlas.

We now use a partition of unity, see Sect. 3.2.

Definition. For ω a k-form with compact support on M^n we set

$$\int_M \omega := \sum_{\alpha\in\mathcal{A}} \int_M \varphi_\alpha \omega.$$

where $\{\varphi_\alpha : \alpha \in \mathcal{A}\}$ is a partition of unity subordinate to an atlas $\{U_\alpha\}$ as before.

This definition is independent of the partition and covering, as

$$\sum_\alpha \int_M \varphi_\alpha\omega = \sum_\alpha \int_M \Big(\sum_\beta \psi_\beta\Big)\varphi_\alpha\omega = \sum_{\alpha,\beta} \int_M \psi_\beta\varphi_\alpha\omega = \sum_\beta \int_M \Big(\sum_\alpha \varphi_\alpha\Big)\psi_\beta\omega = \sum_\beta \int_M \psi_\beta\omega.$$

Note that orientation is implicit in the definition of $\int_M \omega$, and a change of orientation will result in a sign change. Thus, if we were to employ a more precise notation, we would write

$$\int_{(M,\mathcal{S}_+)} \omega = -\int_{(M,\mathcal{S}_-)} \omega.$$

8.2. **Manifolds with boundary.** Let us define the upper half of a ball by

$$B^n_+ := \{x \in \mathbb{R}^n : |x| < 1 \text{ and } x^n \geq 0\}.$$

Note that the bounding $(n-1)$-ball in the plane $\{x^n = 0\}$ is included. We now extend our notion of manifolds to allow for boundary:

Definition. (i) A *topological manifold with boundary* of dimension $n \in \mathbb{N}$ is a topological space M which is Hausdorff, second countable, and such that each point $p \in M$ has a neighbourhood homeomorphic to either B^n or B^n_+. If M has an atlas of such charts with differentiable transition maps then M is a *(differentiable) manifold with boundary.*
(ii) The *boundary* ∂M of a manifold M with boundary is the set of those points $p \in M$ which do not have a neighbourhood homeomorphic to B^n.

The notion of a boundary in (ii) is well-defined since there is no homeomorphism (or diffeomorphism) from B^n_+ onto B^n. Also, ∂M is an $(n-1)$-dimensional manifold of its own, whose charts are given by the restriction of the charts $x\colon U \to B^n_+$ to $x^{-1}\big(B^n_+ \cap \{x^n = 0\}\big)$.

By definition, a manifold in the usual sense can also be considered a manifold with (empty) boundary. It is common to say *closed manifold* to emphasize that a manifold has no boundary, $\partial M = \emptyset$ (nevertheless, considered as a topological space, a manifold with boundary is also closed).

Examples. 1. $\{x \in \mathbb{R}^n : x_n \geq 0\}$
2. $\mathbb{S}^n \cap \{x : x^{n+1} \geq 0\}$.
3. A closed square is not a manifold with boundary since charts cannot be differentiable at the vertices; however the closed square without the four vertices is.
4. If $\psi\colon \mathbb{R}^n \to \mathbb{R}$ with $\operatorname{grad}\psi \neq 0$ on $\psi^{-1}(0)$ then the implicit function theorem gives that $M = \psi^{-1}([0,\infty))$ is a differentiable manifold with boundary $\partial M = \psi^{-1}(0)$.

In order to have $\int_{\partial M} \omega$ well-defined for any $(n-1)$-form ω, we need to define an orientation of ∂M. The idea is simple to explain for the submanifold case: For instance, for $\mathbb{S}^{n-1} = \partial B^n \subset \mathbb{R}^n$ we say the tangent vectors $(v_1, \ldots, v_{n-1}) \in T_p\mathbb{S}^{n-1}$ are positively oriented if the n vectors $(\nu(p), v_1, \ldots, v_{n-1})$ are positively oriented in $\mathbb{R}^n$, where $\nu(p) = p$ is the exterior normal to ∂B^n at p. But a right angle is not defined on a differentiable manifold – this needs the notion of a Riemannian manifold.

So let us instead introduce a substitute for the normal. Note that at a boundary point $p \in \partial M$, we still have the full tangent space $T_pM \sim \mathbb{R}^n$, defined in terms of principal parts $\xi \in \mathbb{R}^n$.

Definition. A tangent vector $v \in T_pM$ at $p \in \partial M$ is *outward pointing* if its principal part w.r.t. to a chart $x\colon U \to B^n_+$ is negative, $dx^n(v) = \xi^n < 0$.

In particular, an outward pointing vector cannot be linearly dependent on any tangent vectors to ∂M, since the latter satisfy $\xi^n = 0$. Thus we can proceed:

Definition. Let M^n be oriented, $p \in \partial M$, and $v_1, \ldots, v_{n-1}$ linearly independent tangent vectors to ∂M. If v is an outward pointing vector at p we define the *induced orientation* at $p \in \partial M$ by

$$\mu_p^{\partial M}(v_1, \ldots, v_{n-1}) := \mu_p^M(v, v_1, \ldots, v_{n-1}).$$

Example. For the half-space $M := \mathbb{R}^n \cap \{x^1 \leq 1\}$ the vector e_1 is an outward pointing normal, and so

$$\mu_p^{\partial M}(e_2, \ldots, e_n) \overset{\text{def}}{=} \mu_p^M(e_1, e_2, \ldots, e_n) = \mu_p^M(e_1, \ldots, e_n). \tag{65}$$

Suppose σ is an orientation preserving (diffeomorphic) singular n-cube in an oriented manifold M, such that its first top face parameterizes a subset of ∂M, that is,

$$\partial M \cap \sigma\big([0,1]^n\big) = \sigma_{1,1}\big([0,1]^{n-1}\big). \tag{66}$$

By (65), the face

$$\sigma_{1,1}\colon [0,1]^{n-1} \to (\partial M, \text{induced orientation})$$

is orientation preserving. Consequently, if ω is an $(n-1)$-form on M with compact support in $\sigma\big((0,1)^n \cup (\{1\} \times (0,1)^{n-1})\big)$ we have $\int_{\sigma_{1,1}} \omega = \int_{\partial M} \omega$. On the other hand, $\sigma_{1,1}$ appears with coefficient $+1$ in $\partial\sigma$ and so altogether

$$\int_{\partial\sigma} \omega = \int_{\sigma_{1,1}} \omega = \int_{\partial M} \omega. \tag{67}$$

We have used the face $\sigma_{1,1}$ (and not $\sigma_{1,0}$ or $\sigma_{n,0}$) in order to have the induced orientation of ∂M and the sign in definition of the boundary operator consistent.

8.3. **Stokes' Theorem for manifolds.**

Theorem 43. *If M is an oriented n-dimensional manifold with boundary ∂M (given the induced orientation), and ω is an $(n-1)$-form with compact support then*

$$\int_M d\omega = \int_{\partial M} \omega.$$

The assumption that ω has compact support is needed to guarantee that the integrals exist. Indeed, for $M = \mathbb{R}$ with $\partial M = \emptyset$ we need an assumption such as f has compact support in order to state $\int_{\mathbb{R}} f' = 0$. In case M itself is compact, the condition is superfluous.

Proof. The manifold M has a countable open cover $\mathcal{O} = \{U_\alpha \subset M \text{ open} : \alpha \in A\}$ with the images of singular orientation preserving diffeomorphic n-cubes which are either interior, or parameterize a subset of the boundary with their first top face as in (66). We let $\{\varphi_\alpha : \alpha \in A\}$ be a partition of unity subordinate to $\mathcal{O}$; as in the previous subsection we require for boundary cubes that the support of φ_α is in $\sigma\big((0,1)^n \cup (\{1\} \times (0,1)^{n-1})\big)$. Finitely many indices suffice to cover the compact set $\operatorname{supp}\omega$.

For each $\alpha \in A$, Stokes' theorem for chains gives

$$\int_{U_\alpha} d(\varphi_\alpha \omega) = \int_{\partial U_\alpha} \varphi_\alpha \omega \overset{(67)}{=} \int_{\partial M} \varphi_\alpha \omega. \tag{68}$$

In case U_α is interior, the function φ_α has no support on ∂M, in which case (68) vanishes.

With sums which are finite at every point we have

$$\sum_{\alpha\in A} d\varphi_\alpha = d\sum_{\alpha\in A} \varphi_\alpha = d\,1 = 0 \quad \Rightarrow \quad \sum_{\alpha\in A} d\varphi_\alpha \wedge \omega = 0 \quad \Rightarrow \quad \sum_{\alpha\in A} \int_M d\varphi_\alpha \wedge \omega = 0.$$

Thus we can sum over (68) to obtain

$$\begin{aligned}\int_{\partial M} \omega &\overset{\text{def.}}{=} \sum_{\alpha\in A} \int_{\partial M} \varphi_\alpha \omega \\ &\overset{(68)}{=} \sum_{\alpha\in A} \int_{U_\alpha} d(\varphi_\alpha\omega) = \sum_{\alpha\in A} \int_{U_\alpha} d\varphi_\alpha \wedge \omega + \varphi_\alpha\, d\omega = \sum_{\alpha\in A} \int_{U_\alpha} \varphi_\alpha\, d\omega \overset{\text{def.}}{=} \int_M d\omega\end{aligned}$$

□

Corollary 44. *If M is orientable, compact and without boundary then $\int_M d\omega = 0$ for all $\omega \in \Lambda^{n-1}M$.*

We have indicated before how the divergence theorem follows from Stokes theorem. Similarly other integral formulas like Green's theorem, the classical Stokes theorem, or the Cauchy integral formula can be derived. A particular case of application of these formulas are the Maxwell equations, which attain an ideal form in the language of differential forms. See problems.

Remark. Arnold [A, Sect.36] uses Stokes' Theorem to define the differential of a form; thereby, he assigns a geometric meaning to the differential. Given an $(n-1)$-form ω on M^n, the differential is the form η to plug into $\int_M \eta = \int_{\partial M} \omega$ to hold. In order to see what η is, consider an infinitesimally small increment to $M^n \subset N^n$ at a point $p \in \partial M$. If ∂M is tangent to $X_1, \dots, X_{n-1}$ and the increment is in a direction X_n at p, then the right hand side of Stokes' formula can be used to compute the formula for $d\omega$. See also the discussion *What is the exterior derivative intuitively?* on the website *mathoverflow*.

8.4. **Hairy Ball Theorem.** We present here only one application of Stokes' theorem in detail.

Two maps between manifolds $f_0, f_1\colon M \to N$ are called (*differentiably*) *homotopic*, if there exists a differentiable map

$$F\colon M \times [0,1] \to N, \quad \text{with } F(x,0) = f_0(x) \text{ and } F(x,1) = f_1(x).$$

As an application of Stokes' Theorem we have:

Lemma 45. *Suppose M^n, N^n are compact orientable manifolds (without boundary). If the maps $f_0, f_1\colon M \to N$ are homotopic then*

$$\int_M f_0^*\omega = \int_M f_1^*\omega \quad \textit{for all } \omega \in \Lambda^n N.$$

Proof. The boundary of the orientable manifold $M \times [0,1]$ is

$$\partial\big(M \times [0,1]\big) = M \times \{1\} - M \times \{0\},$$

where the minus sign denotes opposite orientation. Therefore,

$$\int_M f_1^*\omega - \int_M f_0^*\omega = \int_{\partial(M\times[0,1])} F^*\omega \overset{\text{Stokes}}{=} \int_{M\times[0,1]} d(F^*\omega) = \int_{M\times[0,1]} F^*(d\omega) = 0;$$

the last equality comes from the fact that the $(n+1)$-form $d\omega$ must vanish on the n-dimensional manifold N. $\square$

For the next statement we need the volume form ω of $\mathbb{S}^n \subset \mathbb{R}^{n+1}$. Note that at $p \in \mathbb{S}^n$ the outward normal is also p. Thus we can use the volume form $\det = dx^1 \wedge \cdots \wedge dx^{n+1}$ of $\mathbb{R}^{n+1}$ to define ω:

$$\omega_p(v_1, \dots, v_n) := \det(p, v_1, \dots, v_n) \qquad \text{for } p \in \mathbb{S}^n, v_i \in T_p\mathbb{S}^n$$

We may also define the orientation on $\mathbb{S}^n$ by requiring that this be positive on the vectors v_i.

Although we can get along without an explicit formula for ω let us state that

$$\omega = \sum_{i=1}^{n+1} (-1)^{i-1} x^i dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^{n+1} \ \in \Lambda^n \mathbb{R}^{n+1}. \tag{69}$$

This identity can be verified by determinant development. For simplicity, let us show this specifically for the case $\mathbb{S}^2 \subset \mathbb{R}^3$: Writing $p = (x, y, z)$ and $v, w \in T_p\mathbb{S}^2$, that is, $v, w \perp p$, we develop w.r.t. the first column to obtain (69):

$$\begin{aligned}
\det(p, v, w) &= x \det(e_1, v, w) + y \det(e_2, v, w) + z \det(e_3, v, w) \\
&= x \det \begin{pmatrix} v_2 & w_2 \\ v_3 & w_3 \end{pmatrix} - y \det \begin{pmatrix} v_1 & w_1 \\ v_3 & w_3 \end{pmatrix} + z \det \begin{pmatrix} v_1 & w_1 \\ v_2 & w_2 \end{pmatrix} \\
&\overset{(43)}{=} x\, dy \wedge dz(v, w) - y\, dx \wedge dz(v, w) + z\, dx \wedge dy(v, w)
\end{aligned}$$

Lemma 46. *Suppose n is even. Then the antipodal map $A\colon \mathbb{S}^n \to \mathbb{S}^n$, $A(x) = -x$, is not homotopic to the identity map* id *on $\mathbb{S}^n$.*

Proof. We claim that the volume form ω of $\mathbb{S}^n$ satisfies $A^*\omega = (-1)^{n+1}\omega$. Noting $dA = d(-\operatorname{id}) = -\operatorname{id}$, this can be seen either from the definition of ω, since both p and each of the n vectors v_i changes sign; or likewise from (69) since x^i changes sign under A and so does each of the n differentials dx^j.

Now suppose A was homotopic to the identity. Then, invoking the preceding theorem gives

$$\operatorname{vol}(\mathbb{S}^n) = \int_{\mathbb{S}^n} \omega \overset{\text{Lem.45}}{=} \int_{\mathbb{S}^n} A^*\omega = \int_{\mathbb{S}^n} (-1)^{n+1}\omega = (-1)^{n+1} \operatorname{vol}(\mathbb{S}^n).$$

This is a contradiction for n even. $\square$

Theorem 47 (Hairy Ball Theorem *[Satz vom gekämmten Igel]*)**.** *For n even, each vector field $X \in \mathcal{V}(\mathbb{S}^n)$ has a zero.*

For n odd, however, the number of coordinates of $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ is even and we can consider the vector field $J(p) := (-p_2, p_1, \dots, -p_{n+1}, p_n) \in \mathcal{V}(\mathbb{R}^{n+1})$ which can be considered the rotation by i on $\mathbb{C}^{(n+1)/2} = \mathbb{R}^{n+1}$. Its restriction to $\mathbb{S}^n$ is tangent, $\langle X(p), p\rangle = 0$, and $X \in \mathcal{V}(\mathbb{S}^n)$ does not have a zero.

Proof. Suppose $X(p) \neq 0$ for all $p \in \mathbb{S}^n$. We construct a homotopy $F\colon \mathbb{S}^n \times [0,1] \to \mathbb{S}^n$ from the identity $F(p,0) = p$ to the antipodal map $F(p,1) = -p$, contradicting the preceding lemma. To do so, consider the normalized vector field $X_0 := X/|X|$. Then for all $p \in \mathbb{S}^n$

$$p, X_0(p) \in \mathbb{S}^n \quad \text{with} \quad p \perp X_0(p),$$

and so we can define the desired homotopy by flowing from p to $-p$ along the great circle arc containing the point $X_0(p)$:

$$F(p,t) := \cos(\pi t)\, p + \sin(\pi t)\, X_0(p) \qquad \square$$

Find a homotopy from id to the antipodal map in case n is odd.

8.5. **De Rham cohomology.** We cannot give an appropriate treatment of this topic. Nevertheless, we want to include the definition and point out a few properties.

Recall that a form ω with $d\omega = 0$ is called closed, and if $\omega = d\eta$ it is called exact. Moreover, exact forms are a subset of the closed forms, due to $d^2 = 0$.

Definition. The *de Rham cohomology* groups of a manifold M (perhaps with boundary) are defined as

$$H^k(M) := \frac{\{\text{closed forms in } \Lambda^k M\}}{\{\text{exact forms in } \Lambda^k M\}}\,. \tag{70}$$

In fact, the H^k are vector spaces. Two forms ω_1, ω_2 which belong to the same class in $H^k(M)$, that is $[\omega_1] = [\omega_2]$, are called *cohomologous*; this means that the difference $\omega_1 - \omega_2$ is exact.

By taking pullbacks it is clear that the de Rham cohomology groups agree for diffeomorphic manifolds. It is a deeper fact that they depend on the topology of M alone.

Remark. We have defined $H^k(M)$ for a differentiable manifold M. If M is non-compact then Stokes' Theorem is not applicable to arbitrary forms, but was stated assuming compact support. However, if we restrict to forms with compact support, then the quotient groups (70) obtained will be different from $H^k(M)$. Another issue is the orientability of the manifold, which was also assumed for Stokes theorem, but is not necessary for (70). See [Sp] for a detailed discussion.

Examples. 1. The differential of the angle $d\vartheta$ on $\mathbb{R}^2 \setminus \{0\}$ is a 1-form which represents an element of $H^1(\mathbb{S}^1)$.

2. On the n-torus T^n, the coordinate differential dx^i are well-defined and closed, but not exact. Since these form a basis of 1-forms, we find $H^1(T) = \{\sum a_i dx^i : a_i \in \mathbb{R}\}$ and $H^1(T)$ is n-dimensional. Similarly, any k-fold exterior product of (distinct) dx^i is in $H^k(T)$.

Restricting to basis elements, we see $H^k(T^n)$ has dimension $\binom{n}{k}$.

3. We generalize the first example to the n-sphere to show $H^n(\mathbb{S}^n) \neq 0$. The volume form $\omega \in \Lambda^n \mathbb{S}^n$ can be used to define an n-form $\tilde{\omega}$ on $\mathbb{R}^{n+1} \setminus \{0\}$. Consider the radial projection map $\pi\colon \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{S}^n$, given by $\pi(p) := p/|p|$, and set $\tilde{\omega} := \pi^*\omega$. Then $\tilde{\omega}$ is closed, as $d\tilde{\omega} = d(\pi^*\omega) = \pi^*(d\omega) = 0$ since $d\omega$ is an $(n+1)$-form on the n-dimensional manifold $\mathbb{S}^n$. On the other hand, $\tilde{\omega}$ cannot be exact. Suppose $\tilde{\omega} = d\eta$ and use the inclusion $\iota\colon \mathbb{S}^n \to \mathbb{R}^{n+1}$ to show ω is exact:

$$\omega = \iota^*(\tilde{\omega}) = \iota^*(d\eta) = d(\iota^*\eta)$$

However, as $\int_{\mathbb{S}^{n-1}} \omega = \mathrm{vol}(\mathbb{S}^{n-1}) \neq 0$ this contradicts Corollary 44. For $n = 2$, the form $\omega = (x\,dy - y\,dx)/(x^2 + y^2)$ turns out to be the differential of the angle.

4. For any connected manifold M, the 0-th cohomology is $H^0(M) = \mathbb{R}$. Indeed, exact 0-forms cannot exist (besides 0), so $H^0(M)$ is the vector space of all $f \in C^\infty(M)$ with $df = 0$. On a connected manifold, these functions are constant, and so $H^0(M) = \mathbb{R}$. In general, f is constant on each connected component of M, and so the dimension of $H^0(M)$ is the number of connected components of M.

5. It can be shown $H^n(M) = \mathbb{R}$ for M compact and orientable, and $H^n(M) = 0$ for M compact and non-orientable (see [Sp]).

Definition. A manifold M is *contractible* if there exists a point $p \in M$ there is a homotopy from the identity $f_1 = \mathrm{id}\colon M \to M$ to the constant map $f_0\colon M \to \{p\}$.

Only connected manifolds can be contractible (why?).

If M is compact and contractible and ω is a k-form on M then its pullback with respect to the constant map is $f_0^*\omega(X_1, \ldots, X_k) = \omega_p(0) = 0$. Therefore, by Lemma 45,

$$\int_M \omega = \int_M f_1^*\omega = \int_M f_0^*\omega = 0.$$

Examples. 1. $\mathbb{R}^n$ or any star-shaped subset is contractible.

2. On the other hand, $\mathbb{S}^1$ cannot be contractible, since the angle differential $d\vartheta$ is a form with $\int_{\mathbb{S}^1} d\vartheta = 2\pi \neq 0$.

3. More generally, a compact manifold (without boundary) cannot be contractible: Indeed, these manifolds have a volume form η with $\int_M \eta > 0$, in contradiction to Corollary 44.

The so-called *Poincaré-Lemma* says that on a contractible domain, for instance on $\mathbb{R}^n$, with $n \geq 1$, a closed form is exact (see problems). The intuition here is that path integration along the paths $t \mapsto F(q,t)$ from the marked point p to the arbitrary point q lead to a uniquely defined primitive form: Locally it is the closedness of the form which gives well-definedness, globally it is the contractibility assumption. Thus, for instance, $H^k(\overline{B^n}) = 0$ for all $k > 0$.

A further topic in this context is the mapping degree, in particular a proof of the Brouwer fixed point theorem.

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Part 4. Appendix: Problems

1. Differentiable manifolds and the Whitney embedding theorem

Topological manifolds

Problem 1 – Topological manifolds:

Discuss without proof whether the following sets are topological manifolds. Consider the sets in d) to f) as subsets of $\mathbb{R}^2$ with the standard topology.

a) $M_1 \cap M_2$, b) $M_1 \cup M_2$, c) $M_1 \times M_2$, where M_1, M_2 are topological manifolds,
d) $\{x^2 + y^2 = 1\}$, e) $\{x^2 - y^2 = 1\}$, f) $\{x^2 - y^2 = 0\}$,
g) $(\mathbb{R}^n, \mathcal{O}_1)$ for $\mathcal{O}_1 := \{\text{all subsets of } \mathbb{R}^n\}$, h) $(\mathbb{R}^n, \mathcal{O}_2)$ for $\mathcal{O}_2 := \{\emptyset, \mathbb{R}^n\}$.

Problem 2 – Alexandrow extension:

Let $n \in \mathbb{N}$ and $\mathbb{A}^n := \mathbb{R}^n \cup \{\infty\}$, where $\infty \notin \mathbb{R}^n$. Define open sets by

$$\mathcal{O} := \mathcal{O}_E \cup \mathcal{O}_\infty := \{U \subset \mathbb{R}^n \colon U \text{ is open in } \mathbb{R}^n\} \cup \{\mathbb{A}^n \backslash K \colon K \text{ is compact in } \mathbb{R}^n\}.$$

a) Show that $(\mathbb{A}^n, \mathcal{O})$ is a topological space.
b) Prove that $(\mathbb{A}^n, \mathcal{O})$ is a topological manifold.
c) Show that $\mathbb{A}^n$ is homeomorphic to $\mathbb{S}^n$.

Problem 3 – Non-Hausdorff space:

Let $L := \{0,1\} \times \mathbb{R}$. Define on L an equivalence relation $\sim$ by

$$(0, y_1) \sim (1, y_2) \iff y_1 = y_2 < 0.$$

a) What are the classes on the quotient set $L/\sim$?
b) Show that $L/\sim$ admits a differentiable atlas.
c) Show that $L/\sim$ is not Hausdorff.

Problem 4 – General Linear Group:

We want to show that the general linear group $GL_n(\mathbb{R})$ has exactly two path-connected components.

a) Show that $GL_n(\mathbb{R})$ is not connected. Conclude, that $O(2)$ is not connected.
b) Show that $O(2)$ has exactly two path-connected components.
c) Let Diag_n be the set of regular diagonal $n \times n$-matrices. Show that there is a path connecting $A \in \mathrm{Diag}_n$ to an element of Diag_n consisting only of 1 and -1 entries. Furthermore, prove that if $\det(A) > 0$, then the number of -1 is even.

d) Use that a regular matrix has a LDU decomposition where L is a unit lower triangular matrix, U a unit upper triangular matrix and D is a diagonal matrix to prove that $GL_n(\mathbb{R})$ has exactly two path-connected components.

Differentiable manifolds

Problem 5 – Quiz:

a) Which of the following are differentiable manifolds?: A point; a single cone in $\mathbb{R}^3$; the union of the two coordinate axes in $\mathbb{R}^2$; the closed upper half plane $\{(x, y) \in \mathbb{R}^2 : y \geq 0\}$.
b) Why does any atlas for a compact manifold of dimension $n \geq 1$ contain at least two charts?
c) Describe an atlas for the 2-torus with only two charts.
d) Are an open square and an open disk in the plane diffeomorphic? A sketch suffices to answer.
e) If M, N are manifolds, prove that $M \times N$ are manifolds.
f) Prove that a differentiable manifold has a well-defined dimension.

Problem 6 – Differentiable structures on $\mathbb{R}$*:*

For the topological manifold $\mathbb{R}$, consider the two differentiable atlases

$$\mathcal{A} := \{(\mathrm{id}, \mathbb{R})\}, \qquad \mathcal{B} := \{(x^3, \mathbb{R})\}.$$

a) Verify that x^3 is indeed a chart for $(\mathbb{R}, \mathcal{B})$.
b) Show that the differentiable structures on $\mathbb{R}$ determined by $\mathcal{A}$ and $\mathcal{B}$ are different.
c) Which of the two following maps from $(\mathbb{R}, \mathcal{A})$ to $(\mathbb{R}, \mathcal{B})$ are diffeomorphisms?
   - $f(x) = \sqrt[3]{x}$
   - identity.

Note: There are pairs of differentiable structures that do not arise as a diffeomorphic image of oneanother (see c), for instance on $\mathbb{R}^4$ and many spheres. A non-standard structure is called an *exotic differentiable structure.*

Problem 7 – Differential structure on $\mathbb{R}$*:*

Let $r > 0$ and define $\varphi_r \colon \mathbb{R} \to \mathbb{R}$ by

$$\varphi_r(x) = \begin{cases} x & x \leq 0, \\ rx & x > 0. \end{cases}$$

a) Show that the atlases $\{(\mathbb{R}, \varphi_r)\}_{r>0}$ define an uncountable family of pairwise distinct differentiable structures $\{\varphi_r : r > 0\}$ on $\mathbb{R}$.
b) Prove that the respective differentiable manifolds are pairwise diffeomorphic.

Problem 8 – Two differentiable structures on $\mathbb{R}^2$*:*

Let $M := D = \{q \in \mathbb{R}^2 : |q|^2 < 1\} \subset \mathbb{R}^2$ be the open disk. We consider two charts of M: On the one hand, let $x\colon D \to D$ be the identity. On the other hand, consider a map y of the closed disk to the closed square which preserves the polar angle, and changes the modulus in a way that the bounding $\mathbb{S}^1$ maps to ∂Q, and y is constant speed on rays. That is, after restriction to the open disk,

$$y\colon D \to Q := \{q \in \mathbb{R}^2 \ :\ -1 < q_1, q_2 < 1\} \subset \mathbb{R}^2, \qquad y(q) := \begin{cases} r\left(\frac{q}{|q|}\right)q, & q \neq 0, \\ 0, & q = 0. \end{cases}$$

$$\begin{tikzcd} & M = D \arrow[dl, "x = \mathrm{id}"'] \arrow[dr, "y"] & \\ D \arrow[rr, "y \circ x^{-1} = y"'] & & Q \end{tikzcd}$$

a) Verify that y is a chart for M.
b) Why do x and y each determine a differentiable structure on M?
c) Prove that the two charts x and y are not differentiably compatible. Therefore, the two differentiable structures are not compatible.

Problem 9 – Foliations as non-Hausdorff spaces:

A *foliation [Blätterung]* of $\mathbb{R}^2$ with curves is a decomposition of the entire plane $\mathbb{R}^2$ into a disjoint union of the image of curves $\Gamma_\alpha \in C^1(\mathbb{R}, \mathbb{R}^2)$ where α is element of some index set $\mathcal{F}$. The curves must be injective, regular (i.e., $\Gamma'_\alpha \neq 0$), and have infinite length when restricted to $[0, \infty)$ and $(-\infty, 0]$. That is,

$$\mathbb{R}^2 = \bigcup\{\Gamma_\alpha : \alpha \in \mathcal{F}\}.$$

We call each $L_\alpha := \Gamma_\alpha(\mathbb{R})$ a *leaf* of the foliation.

a) We define a topology on $\mathcal{F}$: A set $U \subset \mathcal{F}$ is open if the union of the leaves represented by U is open in $\mathbb{R}^2$. Convince yourself that this defines a topology.
b) Consider the connected components of two distinct parallel lines in $\mathbb{R}^2$. Foliate the two half-spaces with parallel lines, and the strip inbetween with U-shaped curves (*Reeb foliation*). Show that $\mathcal{F}$ is non-Hausdorff.
c) Increase the number of Reeb components – what does $\mathcal{F}$ look like?
d) Can Reeb components be nested?
Hint: The space in between two Reeb leaves is homeomorphic to an open strip.

Problem 10 – Cylinder:

a) Construct an atlas for the cylinder $\mathcal{C} := \mathbb{S}^1 \times (0, h)$ where $h > 0$.
b) Let $C := [0, r] \times (0, h)$ and define the equivalence relation $(0, y) \sim (r, y)$ on C. Construct an atlas on $C/\sim$.
   Hint: The equivalence relation glues the two opposite sides of the rectangle C together. Visualize it by using a paper sheet.
c) Relate this construction to the one in (a).

Problem 11 – Matrices of fixed rank:

Which of the following three sets are a manifold? All matrices $\{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) : a, b, c, d \in \mathbb{R}\}$ of fixed rank 0, rank 1, or rank 2.
Hint: For rank 1 define a chart on the subset $U_1 := \{M : a \neq 0\}$, etc.

Problem 12 – Stereographic projection:

Let $\mathbb{S}^n := \{x \in \mathbb{R}^{n+1} \colon |x|^2 = 1\}$ be the unit n-sphere, where $|\cdot|$ is the Euclidean norm.

a) Is it possible to construct an atlas on $\mathbb{S}^n$ with only one chart?

Take the 'North pole' $N := (0, 1) \in \mathbb{S}^1$ and define *stereographic projection* $x \colon \mathbb{S}^1 \backslash \{N\} \to \mathbb{R}^1$ such that $x(p)$ is the intersection point of the x-axis with the straight line through N and p.

b) Construct an atlas with the least number of charts on $\mathbb{S}^1$ using the stereographic projection.
c*) Generalize this construction to $\mathbb{S}^n$.

Problem 13 – Properties of stereographic projection:

a) How do the maps $x_\pm$ change, when we project $\mathbb{S}^n \setminus \{\pm N\}$ onto the planes $\mathbb{R}^n \times \{\mp 1\}$ tangent to $\{\pm N\}$?
b) What are the maps $x_\pm$ if we replace $\mathbb{S}^n$ by a sphere of radius $R > 0$?

Problem 14 – Orientability:

Definition: A map $\varphi \colon U \to V$ between subsets of $\mathbb{R}^n$ is called *orientation preserving* if $\det d\varphi > 0$.

a) Is the inversion $x \mapsto -x$ in $\mathbb{R}^n$ orientation preserving?

Definition: Two charts (x, U) and (y, V) of a (differentiable) manifold M have *compatible orientation* if the transition map $y \circ x^{-1} \colon x(U \cap V) \to y(U \cap V)$ is orientation preserving.

b) Do the charts of $\mathbb{R}P^1$ and $\mathbb{R}P^2$ defined in the lecture have a compatible orientation?

Definition: An *atlas* $\mathcal{A} = \{(x_\alpha, U_\alpha) : \alpha \in A\}$ of a manifold M is *oriented* if all its charts have a compatible orientation. A manifold M is *orientable* if it has an oriented atlas $\mathcal{A}$.

c) Check if $\mathbb{R}P^1$ and $\mathbb{R}P^2$ are orientable.
d) Prove that the tangent bundle TM of any differentiable manifold M is orientable.

Definition: An *orientation* of a manifold is a maximal oriented atlas.

e) Prove that a connected orientable manifold has two orientations.
Hint: For any two atlases $\mathcal{A}_1$ and $\mathcal{A}_2$ of M consider

$$s \colon M \to \{\pm 1\}, \qquad s(p) := \operatorname{sign}\bigl(\det d(x_1 \circ x_2^{-1})\bigr)$$

where $(x_i, U_i) \in \mathcal{A}_i$, and $p \in U_i$ for $i = 1, 2$. Show that s is well-defined, that is, independent of the charts chosen and locally constant.

Tangent space

Problem 15 – Quiz:

a) The following curves in $\mathbb{S}^2$ are defined in a neighbourhood of $t = 0$. Which curves are equivalent in $\mathbb{S}^2$ and define the same tangent vector?

$$c_1(t) = \bigl(\cos t, 0, \sin t\bigr) \qquad c_2(t) = \bigl(\sin t, 0, \cos t\bigr)$$
$$c_3(t) = \bigl(\cos(2t), 0, \sin(2t)\bigr) \qquad c_4(t) = \bigl(\sqrt{1-t^2}, 0, t\bigr)$$

b) Let x be a chart of M^n at p, and $\xi, \eta \in \mathbb{R}^n$. Which of the following curves, defined for t in a neighbourhood of 0, represent the same tangent vector?

$$x^{-1}\bigl(x(p) + t\xi\bigr) \qquad x^{-1}\bigl(x(p) + \sin t\,\xi + \cos t\,\eta\bigr)$$
$$x^{-1}\bigl((1+t^2)x(p) + t\xi\bigr) \qquad tx^{-1}\bigl(x(p) + \xi\bigr)$$

Problem 16 – Tangent vectors to $\mathbb{R}P^2$:

Consider the point $p = [1, 1, 0] \in \mathbb{R}P^2$ and the charts x_1 and x_2 given in the lecture.

a) Find curves $c_1(t)$, $c_2(t)$ in $\mathbb{R}^2$ which represent the standard basis at p w.r.t. x_1.
b) Decide if c_1, c_2 also represent the standard basis w.r.t. x_2. To do so, consider the representing curves $d_i(t) := (x_2 \circ x_1^{-1})(c_i(t))$ in the image of x_2.
c) Which linear mapping maps $c_i'(0)$ to $d_i'(0)$?

Problem 17 – Tangent space:

a) How did we define a tangent vector $v \in T_pM$ to a manifold M? What is the standard basis of T_pM with respect to a chart (x, U)?
b) Consider an implicitly defined submanifold $M = \varphi^{-1}(0)$, where φ has 0 as a regular value. Describe its tangent space.

c) If y is a chart which locally maps a submanifold $M \subset \mathbb{R}^{n+k}$ to a slice, i.e. $y(M \cap U) = y(U) \cap (\mathbb{R}^n \times \{0\}) \subset \mathbb{R}^n \times \mathbb{R}^k$, where $U \subset \mathbb{R}^{n+k}$, how would you describe the tangent space of M at $p \in M$?

d) Show that the manifold TM is Hausdorff.

Problem 18 – Tangent space for products and graphs:

Let M, N be differentiable manifolds, $\pi\colon M \times N \to N$ be the projection map, and for $q \in N$ let $f^q\colon M \to M \times N$ be a differentiable injection defined by $p \mapsto (p, q)$. Prove that

a) $T_{(p,q)}(M \times N) = T_pM \times T_qN$,

b) $d\pi_{(p,q)}\colon T_pM \times T_qN \to T_pM$ is also the projection map,

c) f^q is a diffeomorphism onto its image and $d(f^q)_p\colon v \mapsto (v, 0)$,

d) if $f\colon M \to M'$ and $g\colon N \to N'$ are differentiable, then $d(f \times g)_{(p,q)} = df_p \times dg_q$.

Let $h\colon M \to N$ be a differential map and define $H\colon M \to M \times N$ by $p \mapsto (p, h(p))$. Prove that

e) $dH_p(v) = (v, dh_p(v))$, and hence $T_{(p,h(p))}\,\mathrm{graph}(h)$ is the graph of $dh_p\colon T_pM \to T_{f(p)}N$.

Problem 19 – Torus:

Prove that following three spaces are homeomorphic:

a) The subspace $T_1 := \{p \in \mathbb{R}^3\colon d(p, S) = r\}$ for $0 < r < 1$ and $S := \{(x, y, 0) \in \mathbb{R}^3\colon x^2 + y^2 = 1\}$. Here T_1 is equipped with the subspace topology.

b) The product space $T_2 := \mathbb{S}^1 \times \mathbb{S}^1$ equipped with the product topology.

c) The quotient space $T_3 := \big([-1, 1] \times [-1, 1]\big)/\sim$ equipped with the quotient topology where the equivalence relation given by $(s, -1) \sim (s, 1)$ and $(-1, t) \sim (1, t)$.

Problem 20 – Physicist's definition of tangent space:

Let $(M, \mathcal{S})$ be a differentiable manifold of dimension n. For $p \in M$ consider the charts containing $p \in M$,

$$\mathcal{X}(p) := \{(x, U) \in \mathcal{S} : p \in U\}.$$

We define a relation $\sim$ on $\mathcal{X}(p) \times \mathbb{R}^n$ via

$$\big((x, U), \xi\big) \sim \big((y, U'), \eta\big) \quad :\Longleftrightarrow \quad \eta = d(y \circ x^{-1})\big|_{x(p)}\xi.$$

The set of equivalence classes is $T_p^{\mathrm{phys}}M := \big\{ \big[((x, U), \xi)\big] : (x, U) \in \mathcal{X}_p \big\}$.

a) Confirm that $\sim$ defines an equivalence relation.

b) Prove: For $(x, U) \in \mathcal{X}_p$, the mapping

$$T_pM \ni [c] \mapsto \left[\left((x, U), \frac{d}{dt}(x \circ c)\big|_0\right)\right] \in T_p^{\text{phys}}M$$

- is well-defined,
- independent of the choice of chart $(x, U) \in \mathcal{X}_p$,
- bijective.

Remark: There is a further common definition of the tangent space: A *derivation* at the point p is a linear map $D\colon C^\infty(M) \to \mathbb{R}$ satisfying the Leibnitz rule $D(fg) = f(p)D(g) + g(p)D(f)$. In fact a tangent vector is a derivation at p, as we can identify $[c]$ with the derivation given by $f \mapsto (f \circ c)'(0)$.

Problem 21 – Cotangent Space:

For each point p of a manifold M we want to define the *cotangent space* T_p^*M. One way to do this is in terms of functions: A *cotangent vector to* M *at* $p \in M$ is an equivalence class of functions $f \in C^\infty(U)$ where $U \ni p$ is open, under the relation

$$f \sim g \quad :\Leftrightarrow \quad df_p = dg_p.$$

a) Show that this is well-defined.
b) Show that T_p^*M is a vector space and construct a basis for T_p^*M. What is the dimension of the cotangent space?
c) Show that the cotangent space is the dual vector space of the tangent space.
d) Define the *cotangent bundle* T^*M, an atlas for the cotangent bundle and show that T^*M is a $2n$-dimensional manifold.

Remark: There are other definitions of the cotangent space. Clearly, another characterisation of $f \sim g$ is $f - g \in \{h \in C^\infty(M)\colon dh|_p = 0\}$. Then setting $\mathfrak{m}_p = \{h \in C^\infty(M)\colon h(p) = 0\}$ and noting

$$\mathfrak{m}_p^2 = \left\{\sum_{i=1}^k f_i g_i \in C^\infty(M) : k \in \mathbb{N} \text{ and } f_i, g_i \in \mathfrak{m}_p\right\} = \{h \in \mathfrak{m}_p : dh|_p = 0\},$$

we obtain $T_p^*M = \mathfrak{m}_p/\mathfrak{m}_p^2$, which is the usual definition in algebraic geometry. Moreover, the tangent space T_pM can be defined as the dual of T_p^*M.

Problem 22 – Vector bundles:

A *vector bundle* is a triple (E, M, π_E) subject to the following. First, the map $\pi_E\colon E \to M$ is surjective, second for each $p \in M$ the *fibre* $E_p = \pi_E^{-1}(p)$ has the structure of a vector space, and third for all $p \in M$ there exists an open neighbourhood $U \subset M$ and a diffeomorphism $\varphi\colon \pi_E^{-1}(U) \to U \times \mathbb{R}^n$ such that for all $q \in U$ the restriction $\varphi|_{E_q}$ is a vector space isomorphism. We also say E is a vector bundle over M.

a) Show that the tangent bundle of a differentiable manifold M is a vector bundle over M.

Definition: A vector bundle (E, M, π_E) is *trivial* if $E = M \times \mathbb{R}^k$ where $\pi_E(p, v) = p$. For instance, the cylinder $\mathbb{S}^1 \times \mathbb{R}$ is trivial. In contrast, a non-trivial vector bundle is the Möbius strip, which is a product $I \times \mathbb{R}$ only locally.

b) Show a vector bundle (E, M, π_E) is trivial if and only if there are maps $s_1, \ldots, s_n \colon M \to E$ such that $\pi_E \circ s_i = \mathrm{id}_M$ and $\{s_1(p), \ldots, s_n(p)\}$ is a basis for all $p \in M$.

Let G be a Lie group and for $p \in G$ let $L_p \colon G \to G$ be (differentiable) left multiplication, i.e., $L_p(q) := pq$.

c) Prove that the tangent bundle of a Lie group is trivial.
d) Conclude that the tangent bundle of $\mathbb{S}^1$ and $T_2 := \mathbb{S}^1 \times \mathbb{S}^1$ are trivial.

Problem 23 – Framing of the 3-sphere:

The goal is to prove with simple calculations that on the 3-sphere

$$\mathbb{S}^3 := \{x \in \mathbb{R}^4 : x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1\}$$

there are three unit vector fields which are orthonormal. Thus the tangent bundle $T\mathbb{S}^3$ is diffeomorphic to $\mathbb{S}^3 \times \mathbb{R}^3$.

a) Define three $\mathsf{SO}(4)$-matrices, which in shorthand notation ($\mathsf{SU}(2)$) are

$$I := \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad J := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad K := \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}.$$

Here, i denotes the 2×2 block matrix $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ and 1 the identity. Prove the relations $IJ = K = -IJ$, $JK = I = -KJ$, $KI = J = -IK$ and $I^2 = J^2 = K^2 = -E$, where E is the 4×4 identity matrix.

b) Let $\mathcal{M} := \operatorname{span}\{E, I, J, K\}$ be a four-dimensional subspace of the 4×4-matrices. We consider the vector space isomorphism

$$\Phi \colon \mathbb{R}^4 \to \mathcal{M}, \qquad (x_1, x_2, x_3, x_4) \mapsto x_1 E + x_2 I + x_3 J + x_4 K,$$

and define a multiplication on $\mathbb{R}^4$ in terms of the matrix product,

$$\cdot \colon \mathbb{R}^4 \times \mathbb{R}^4 \to \mathbb{R}^4, \qquad x \cdot y := \Phi^{-1}\big(\Phi(x)\, \Phi(y)\big).$$

Prove that this non-commutative multiplication is well-defined and makes the vector space $\mathbb{R}^4$ into a division ring *[Schiefkörper]* $\mathbb{H}$, called *quaternions*. The letter $\mathbb{H}$ honours William Hamilton (1805–1865).

c) Show that for each of the three matrices M, we have $Mx \perp x$ for all $x \in \mathbb{S}^3$ (w.r.t. the real scalar product on $\mathbb{R}^4$!). That is, $Mx \in T_xM$ and so $x \mapsto Mx$ is a vector field on $\mathbb{S}^3$, called *Hopf vector field*.

d) Prove that the three vector fields Ix, Jx, Kx are orthonormal at each $x \in \mathbb{S}^3$.

e) Show that the integral curves of the three vector fields are great circles in $\mathbb{S}^3$, i.e., for all $x \in \mathbb{S}^3$, the great circles $c(t) := x\cos t + Mx\sin t$ have tangent vector $Mc(t)$.

Problem 24 – Unit quaternions $\mathbb{S}^3 \subset \mathbb{H}$ *parameterize* $\mathsf{SO}(3)$*:*

The quaternions $\mathbb{H}$ are the vector space $\mathbb{R}^4$ with the following non-commutative multiplication. On a basis denoted by $\{1, i, j, k\}$ the product is the $\mathbb{R}$-linear extension of

$$i^2 = j^2 = k^2 = -1, \qquad ijk = -1$$

as well as by $1u = 1$ for all $u \in \mathbb{R}^4$. In case you solved Problem 11, our notation is $1 := \Phi^{-1}(E)$, $i := \Phi^{-1}(I)$, etc.

We define a *conjugation* map $\bar{\cdot}\colon \mathbb{H} \to \mathbb{H}$ as the $\mathbb{R}$-linear map satisfying

$$\bar{1} = 1, \quad \bar{u} = -u \quad \text{ for } u \in \{i, j, k\}.$$

Moreover we set

$$\operatorname{Re} x := \frac{x + \bar{x}}{2}, \quad \text{ and } \quad \operatorname{Im} x := x - \operatorname{Re} x.$$

We consider $\operatorname{Re} x$ a real number, and we identify $\operatorname{Im}\mathbb{H} = \mathbb{R}^3$.

a) Prove $\overline{xy} = \bar{y}\bar{x}$ and $x\bar{x} = \bar{x}x = |x|^2$, where the norm is induced by the standard scalar product on $\mathbb{R}^4$ which makes $\{1, i, j, k\}$ an orthonormal basis.
b) Use a) to prove $|xy| = |x||y|$ and conclude the *unit quaternions* $\mathbb{S}^3 = \{x \in \mathbb{H} : |x| = 1\}$ form a group.
c) Prove $y^2 = -|y|^2$ if and only if $y \in \operatorname{Im}\mathbb{H}$, that is, $y = \operatorname{Im} y$. For $x \in \mathbb{S}^3$ we now define $K_x\colon \mathbb{H} \to \mathbb{H}$ by $K_x(y) := xyx^{-1}$. Then conclude $K_x(\operatorname{Im}\mathbb{H}) = \operatorname{Im}\mathbb{H}$. Why is K_x norm preserving (or isometric) on $\operatorname{Im}\mathbb{H} = \mathbb{R}^3$? Assert that in fact $K_x \in \mathsf{SO}(3)$.
d) We now want to prove that $x \mapsto K_x$ is a continuous group homomorphism from $\mathbb{S}^3$ to $\mathsf{SO}(3)$, such that $K_x = K_\xi$ holds only for $x = \pm\xi$. This proves that $\mathsf{SO}(3)$ is homeomorphic to $\mathbb{R}P^3$. Check first $K_x = K_{-x}$ and that K is a group homomorphism. Then prove that $\operatorname{Im} x \in \mathbb{R}^3$ spans the axis of the rotation K_x (defined for $x \neq \pm 1$). To determine the angle ϑ_x of the rotation check the case $x = a + bi$, $y = j$, and use it to recover the general case.

Remarks 1. This is the most efficient parameterization of the rotation group of Euclidean space, used in mathematics and computer graphics. The *Euler angles* are another choice, but lead to a more involved composition rule.
2. There are nice topological consequences: As experiments indicate, the group $\mathsf{SO}(3) = \mathbb{R}P^3$ is not simply connected: Its double cover is $\mathbb{S}^3$. (See Berger, Geometry I, Sect. 8.9.)

Differentiable maps and submanifolds

Problem 25 – Immersions and embeddings:

Discuss whether the following curves are immersions, injective immersions, or embeddings:

a) $c_1\colon \mathbb{R} \to \mathbb{R}^2,\ c_1(t) = (t, |t|)$,
b) $c_2\colon (-\pi, \pi) \to \mathbb{R}^2,\ c_2(t) = (\cos(t), \sin(t))$,
c) $c_3\colon \mathbb{R} \to \mathbb{R}^2,\ c_3(t) = (\cos(t), \sin(t))$,
d) $c_4\colon (0, \infty) \to \mathbb{R}^2,\ c_4(t) = \left(\frac{1}{t}\cos(t), \frac{1}{t}\sin(t)\right)$,
e) $c_5\colon (0, \infty) \to \mathbb{R}^2,\ c_5(t) = \left(\frac{t}{1+t}\cos(t), \frac{t}{1+t}\sin(t)\right)$.

Problem 26 – Transversality:

Let M and N be submanifolds of a manifold Y. We say M and N are *transverse* if

$$T_pM + T_pN = T_pY \qquad \text{at each point } p \in M \cap N.$$

Furthermore, we say a differentiable map $f\colon M \to Y$ is *transverse* to a submanifold Z of Y, if

$$df_q(T_qM) + T_{f(q)}Z = T_{f(q)}Y \qquad \text{for each } q \in f^{-1}(Z).$$

a) Let M, N be a plane or a line in $\mathbb{R}^3$. When are M and N transverse?
b) Consider $M := \mathbb{R}^k \times \{0\}$ and $N := \{0\} \times \mathbb{R}^\ell$ in $\mathbb{R}^n$. When are M and N transverse?
c) Prove: If $f\colon M \to Y$ is transverse to Z then $f^{-1}(Z)$ is a submanifold of Y.
d) Moreover, then the codimension of $f^{-1}(Z)$ in M is equal to that of Z in Y.
e) Conclude: If M and N are transverse in some ambient manifold Y then the intersection $M \cap N$ is a manifold.
f) Let f be a differentiable map from a compact manifold M to a manifold Y. Show that the transversality of f with respect to given submanifold $Z \subset Y$ is stable under small deformation of f, i.e., for every differentiable homotopy $F\colon M \times [0,1] \to Y$ of $F_0 := F(\cdot, 0)$ there is an $\varepsilon > 0$ such that for all $0 \le s < \varepsilon$, the map $F_s := F(\cdot, s)$ is also transverse to Z.

Problem 27 – Quiz:

a) Find examples of a holomorphic maps $f\colon U \to V$, with U, V suitable domains in $\mathbb{C}$, which represent the following, when considered as real differentiable maps:
   - A local diffeomorphism which is not a diffeomorphism.
   - An embedding which is not a diffeomorphism.
b) If two manifolds M and N have the same dimension n, then which of the following notions agree for maps from M to N?
   • immersion • embedding • local diffeomorphism • diffeomorphism onto its image
c) Define the subspace topology of a subset Y of a topological space X and prove it defines a topology.

Problem 28 – Vanishing differential:

Suppose $f\colon M \to N$ is differentiable such that $df_p\colon T_pM \to T_pN$ vanishes for all $p \in M$. Prove that f is constant on connected components.

Problem 29 – Submersions:

A differentiable map $f\colon M^m \to N^n$ is a *submersion* if df_p is surjective for all $p \in M$.

a) Give an example of a submersion where one of m, n equals 2, and the other equals 1. State the relationship between m and n in general.
b) Prove that a submersion is an open mapping, that is, for $U \subset M$ open the image $f(U)$ is open. *Hint:* Adapt the proof of Theorem 8.
c) Let M^n be compact. Prove there is no submersion $f\colon M \to \mathbb{R}^n$.

Problem 30 – Proper maps:

A continuous mapping $f\colon M \to N$ between topological manifolds M and N is *proper [eigentlich]* if each compact set $K \subset N$ has a compact preimage $f^{-1}(K)$.

a) Give an example of a proper and a non-proper map from $\mathbb{R}$ to $\mathbb{R}$.
b) Prove that a curve $c\colon \mathbb{R} \to \mathbb{R}^n$ is proper if and only if $\lim\limits_{t\to\pm\infty} \|c(t)\| = \infty$.
c) Let $f\colon M \to N$ be continuous and let M be compact. Show that f is proper.
d) Check the following maps for properness:
   - $f_1\colon \mathbb{R} \to \mathbb{R},\ p \mapsto p^3$,
   - $f_2\colon (-\pi, \pi) \to \mathbb{R}^2,\ f_2(t) = (\cos(t), \sin(t))$,
   - $f_3\colon \mathbb{S}^1 \to \mathbb{R}^2,\ p \mapsto (p_1,\, p_1 p_2)$.

Problem 31 – Idempotent maps and submanifolds:

Let M be a differentiable manifold and let $f\colon M \to M$ be differentiable and *idempotent*, i.e. $f \circ f = f$.

a) Give an example of an idempotent map, besides the identity, linear projections and constant functions.
b) Show that $f(M) = \{p \in M\colon f(p) = p\}$ is closed and if M is connected, then $f(M)$ is also connected.
c) Prove that $\operatorname{im}(df_q) = \ker(\mathrm{id}_{T_qM} - df_q)$ for all $q \in f(M)$.
d) Show that if $\operatorname{rank}(df_p) = r$ that there exist a neighborhood U such that $\operatorname{rank}(df_p) \geq r$ for all $q \in U$.
   Remark: In fact, this is true in general: The rank of a mapping is a lower semicontinuous function.
e) Conclude that for M connected $p \mapsto \operatorname{rank}(df_p)$ is constant on $f(M)$.
f) Assuming again M is connected, prove that $f(M) \subset M$ is a submanifold of dimension $\operatorname{rank}(df_p)$ for any $p \in M$.

Problem 32 – Helicoids in $\mathbb{S}^3$:

Let $c \in \mathbb{R}$ be a parameter and consider the mapping

$$h = h_c\colon \mathbb{R}^2 \to \mathbb{S}^3 \subset \mathbb{R}^4, \qquad (u,v) \mapsto \begin{pmatrix} \cos u \cos v \\ \cos u \sin v \\ \sin u \cos(cv) \\ \sin u \sin(cv) \end{pmatrix}.$$

a) h_c is an immersion of $\mathbb{R}^2$ for $c \neq 0$.
Hint: Calculate the 2×2 minor of the Jacobian $Jh = \left(\frac{\partial}{\partial u}h, \frac{\partial}{\partial v}h\right)$.

b) Show the two *axes* $v \mapsto a_1(v) = h(0,v)$ and $v \mapsto a_2(v) = h(\frac{\pi}{2}, v)$ are great circles whose points are pairwise perpendicular. Identifying $\mathbb{R}^4$ with $\mathbb{C} \times \mathbb{C}$, how would you write a_1 and a_2?

c) The maps $u \mapsto h(u,v) = (\cos u)a_1(v) + (\sin u)a_2(v)$ parameterize great circles with unit speed, and these circles meet the two axes at right angles. (What does it mean for two curves to meet at a right angle?) In this sense, h represents a helicoid in $\mathbb{S}^3$.

d) Try to identify the image surface for $c = 0$. What is the subset of $\mathbb{R}^2$ where h_0 fails to be an immersion and what is its image?

e) Consider $c = 1$. Determine a maximal set $\Omega \subset \mathbb{R}^2$ such that h_1 is injective.
Hint: Verify first $h_1(u + 2\pi, v) = h_1(u, v + 2\pi) = h(u,v)$; you need, however, one more relation to determine Ω. One way to find it would be to consider all intersection points of the two great circles $u \mapsto h_1(u,0)$ and $v \mapsto h_1(0,v)$.

Problem 33 – Klein bottle (continues previous problem):

The Klein bottle K is the set $[0,1] \times [0,1]$, where opposite edges are identified (equivalence relation!) as follows: One pair of opposite edges in the same direction, the other in opposite directions.

a) Prove that K is a 2-manifold by defining a basis for the topology and charts.

b) Prove that the helicoid h_2 (or $h_{1/2}$) represents a Klein bottle immersed in $\mathbb{R}^4$. To do so, determine again minimal *periods* for h as in the previous problem.

c) Does h_2 represent an embedding of the Klein bottle into $\mathbb{S}^3$?

Remarks: 1. Since the Klein bottle is complete (as a metric space, say) and non-orientable, it cannot be embedded into $\mathbb{R}^3$. Indeed, any such embedding can be shown to divide space into two components, and the normal direction pointing into one of these components contradicts the non-orientability.
2. `en.wikipedia.org/wiki/Klein_bottle#4-D_non-intersecting` states an embedding of the Klein bottle into $\mathbb{R}^4$.

Problem 34 – Veronese Embedding:

Define the map

$$f\colon \mathbb{S}^2 \to \mathbb{R}^6, \quad (x,y,z) \mapsto (x^2, y^2, z^2, \sqrt{2}\,xy, \sqrt{2}\,yz, \sqrt{2}\,zx).$$

a) Prove that f is an immersion.

b) Show that $f(\mathbb{S}^2)$ is contained in $\mathbb{S}^5 \subset \mathbb{R}^6$.

c) Prove that f is contained in a hyperplane of $\mathbb{R}^6$. Conclude that f can be assumed to take an image in $\mathbb{S}^4_r$ for some $0 < r < 1$.

d) Prove that $f(\mathbb{S}^2)$ is a proper subset of $\mathbb{S}^4_r$. Compose f with a suitable map to find an immersion $F\colon \mathbb{S}^2 \to \mathbb{R}^4$.

The real projective space $\mathbb{R}P^n$ can be identified with the quotient space $\mathbb{S}^n/\sim$ where $\sim$ identifies antipodal points (i.e. $x \sim -x$). The quotient topology then agrees with the topology of $\mathbb{R}P^n$.

e) Show that F induces an embedding of $\mathbb{R}P^2$ into $\mathbb{R}^4$, the *Veronese embedding.*

f) Can you generalize the construction to obtain an embedding of $\mathbb{R}P^n$ into some $\mathbb{R}^k$? Compare k with the optimal dimension predicted by the Whitney theorem.

Remark: Here is the geometric intuition for the definition of the Veronese embedding. A point $p \in \mathbb{R}P^2$ is a line in $\mathbb{R}^3$. It is natural to associate to p the orthogonal projection of $\mathbb{R}^3$ onto the line p, which is a linear map Π_p for each $p \in \mathbb{R}P^2$. Explicitly, Π_p is given by $\Pi_p(x,y,z) = \langle (x,y,z), u/|u|\rangle u/|u|$, where $u \in \mathbb{R}^3 \setminus \{0\}$ represents the line p. The linear map Π_p is represented by a symmetric 3×3-matrix A_p. Let us denote the space of these matrices by $\mathrm{Sym}(3) \subset \mathbb{R}^9$. Thus there is a map $\tilde{f}\colon \mathbb{R}P^2 \to \mathbb{R}^9$. As a $(*)$-problem, relate $\tilde{f}$ to f.

Problem 35 – Grassmannians:

We consider

$$G(k,n) := \{k-\text{dimensional subspaces } V \subset \mathbb{R}^n\}.$$

We want to prove that $G(k,n)$ is a manifold with a suitable differentiable structure.

a) Consider $\mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^{n-k}$. Prove that $U := \{V \in G(k,n) : V \cap (\{0\} \times \mathbb{R}^{n-k}) = \{0\}\}$ is a manifold by regarding U as the set of graphs $\Gamma(A)$ of linear mappings $A\colon \mathbb{R}^k \to \mathbb{R}^{n-k}$. What is the dimension? (Perhaps the same works implicitly.)

b) Find charts on sets similar to U that cover $G(k,n)$.

c) Show that the transition maps are differentiable (this is harder).

d) Find a bijection from $G(k,n)$ to $G(n-k,n)$. Is it a diffeomorphism?

Problem 36 – Continuous injections of compact spaces are homeomorphisms:

Definition: A map $f\colon X \to Y$ between topological spaces is a *closed map* if each closed subset $A \subset X$ has a closed image $f(A) \subset Y$.
Prove the following topological lemma:
If $f\colon X \to Y$ is a continuous map of topological spaces, where X is compact and Y is Hausdorff, then f is closed.

Give the proof in three steps:

a) A closed subset A of a compact space X is compact itself.
b) The continuous image $B := f(A)$ of a compact set A is again compact.
c) A compact subset B of a Hausdorff space Y is closed.

The section *Some topology* of the course notes explains why this lemma is the essential step to prove the statement in the problem title.

Whitney theorem

Problem 37 – Immersions and embeddings:

a) In its improved form, the Whitney embedding theorem states that for $n \geq 2$ each n-manifold can be immersed into $\mathbb{R}^{2n-1}$, and embedded into $\mathbb{R}^{2n}$. Discuss these statements for the case $n = 1$.
b) Can a Möbius strip be embedded into $\mathbb{R}^3$?
c) Find a 2-manifold which cannot be embedded into $\mathbb{R}^3$ (and reason for the fact).

Problem 38 – Klein Bottle:

The *Klein bottle* is a non-orientable two-dimensional surface. Like the torus, it can be defined by identifying edges of a square. Trying to immerse the bottle into three-dimensional space results in self-intersections; see Figure 1 and `en.wikipedia.org/wiki/Klein_bottle` for further pictures.

However, the Whitney embedding theorem predicts we can embed the Klein bottle into four-dimensional space. To see how, let us quote from the book of Guillemin and Pollack: ‘We can envision an embedding in $\mathbb{R}^4$, represent the fourth dimension by density of red coloration and allow the bottle to blush as it passes through itself.’

Why can the two leaves of the self-intersection set indeed carry a distinct colouring? Use the above pictures and discuss the existence of a colouring on the preimage as rigorous as you can.

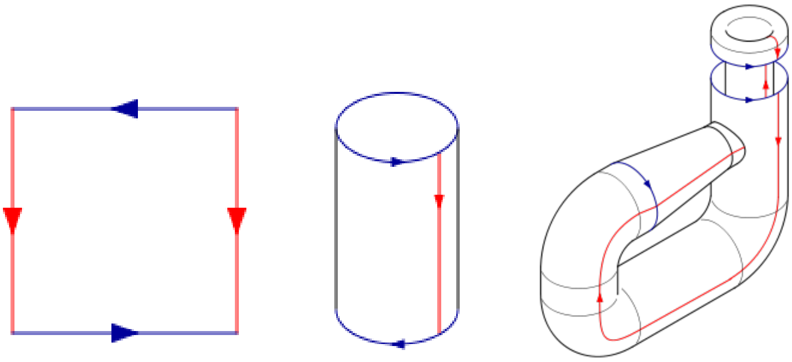

FIGURE 1. Identifying the edges of a square as indicated gives a Klein bottle, which can only be realized in $\mathbb{R}^3$ as an immersion.

Problem 39 – Whitney embedding obtained by projection:

Consider a compact submanifold $M^n \subset \mathbb{R}^N$ where $N > 2n+1$. We show M admits an embedding in $\mathbb{R}^{2n+1}$. To do so, prove: If $f\colon M \to \mathbb{R}^N$ is an injective immersion then there exists a unit vector $a \in \mathbb{R}^N$ such that the composition of f with the projection map π carrying $\mathbb{R}^N$ onto the orthogonal complement of a is injective and an immersion.

Problem 40 – Submanifolds as metric spaces:

Let $f\colon M \to \mathbb{R}^{n+k}$ be an embedding of a connected manifold M. Prove that the following define metrics on M:

$$d_1(p,q) := |f(p) - f(q)|, \quad d_2(p,q) := \inf\{L(f\circ c) : c \in PC^1([0,1],M),\, c(0)=p,\, c(1)=q\}$$

Here $PC^1([0,1],M)$ stands for continuous maps which are *piecewise* C^1, that is, they are C^1 when restricted to $[t_i, t_{i+1}]$, where $0 = t_0 < t_1 < \ldots < t_k = 1$ with $k \in \mathbb{N}$. A curve in PC^1 has the length $L(c) := \sum_{i=1}^k L(c|_{[t_{i-1},t_i]})$.

Is the same true if
- f is only an immersion, or if
- M is disconnected, i.e. M is a union of manifolds?

Problem 41 – A set of measure 0 has a continuous image with positive measure:

Let $Q := [0,1] \times [0,1]$ be the square in the plane $\mathbb{R}^2$. A space-filling curve is a curve $c\colon [0,1] \to Q$ which is continuous and surjective. Use this example to construct a continous mapping $f\colon Q \to Q$ which maps a set of measure 0 to a set of positive measure.

Problem 42 – Space Filling Curve:

Define $f\colon \mathbb{R} \to \mathbb{R}$ by following properties: f is even $(f(-t) = f(t))$, 2-periodic $(f(t+2) =$

$f(t))$, and f satisfies

$$f(t) = \begin{cases} 0 & \text{if } 0 \le t \le \frac{1}{3}, \\ 3t-1 & \text{if } \frac{1}{3} < t < \frac{2}{3}, \\ 1 & \text{if } \frac{2}{3} \le t \le 1. \end{cases}$$

Set

$$x(t) = \frac{1}{2} \sum_{k=0}^{\infty} \frac{f(3^{2k}t)}{2^k}, \qquad y(t) = \frac{1}{2} \sum_{k=0}^{\infty} \frac{f(3^{2k+1}t)}{2^k}.$$

The curve $\gamma\colon [0,1] \to [0,1]^2$, $\gamma(t) = (x(t), y(t))$ is called *Schoenberg curve.*

a) Prove $\gamma(t) \in [0,1]^2$ for all $t \in [0,1]$.
b) Show that γ is continuous. *Hint:* uniform convergence.
c) Prove that γ is surjective.
   Hint: For $(x_0, y_0) \in [0,1]^2$ use a binary representation

$$x_0 = \sum_{k=0}^{\infty} \frac{a_k}{2^{k+1}},\ a_k \in \{0,1\}, \qquad y_0 = \sum_{k=0}^{\infty} \frac{b_k}{2^{k+1}},\ b_k \in \{0,1\}.$$

   Define $(c_k)_{k\in\mathbb{N}}$ by $c_{2k} := a_k$ and $c_{2k+1} := b_k$. Then set

$$t_0 := \sum_{k=0}^{\infty} \frac{2c_k}{3^{k+1}}.$$

   and check $0 \le t_0 \le 1$. Finally, show $\gamma(t_0) = (x_0, y_0)$.

Problem 43 – Partition of Unity in $\mathbb{R}^2$:

a) Let $B_r(m,n) = \{(x,y) \in \mathbb{R}^2 : (x-m)^2 + (y-n)^2 < r^2\}$ for $(m,n) \in \mathbb{Z}^2$ and $r \ge 1$. Sketch some sets $B_1(m,n)$. Why do they form an open covering of $\mathbb{R}^2$?
b) Sketch the graph of the function

$$\psi \in C^\infty(\mathbb{R}, [0,1]), \qquad \psi(t) := \begin{cases} e^{-\frac{1}{1-t^2}} & \text{for } t \in (-1,1), \\ 0 & \text{otherwise.} \end{cases}$$

c) Use the function ψ to construct a partition of unity subordinate to the covering $\{B_r(m,n) : m,n \in \mathbb{Z}\}$ of $\mathbb{R}^2$ for a suitable value of r.

Problem 44 – Submanifolds of Euclidean space:

Let $f\colon \mathbb{R}^m \to \mathbb{R}^k$ be a differentiable map. A point $q \in \mathbb{R}^k$ is a *regular value of f* if for all $p \in M := f^{-1}(q)$ the map df_p is surjective. Recall that if q is a regular value, then M is a submanifold of $\mathbb{R}^m$, and the tangent space of M at p is equal to $\ker(df_p)$; moreover the codimension of M is equal to k.

a) Prove that the set $M := \{p \in \mathbb{R}^3 \colon p_1^3 + p_2^3 + p_3^3 = 1,\ p_1 + p_2 + p_3 = 0\}$ is a differentiable manifold. What is the dimension of M?
b) Prove that $O(n)$ is a submanifold of dimension $n(n-1)/2$ and show that the tangent space of $O(n)$ at the identity E is the space of skew-symmetric matrices, i.e., prove that $T_E O(n) = \{A \in M_n(\mathbb{R}) \colon A^T + A = 0\}$.
Hint: Consider the map $f \colon \mathbb{R}^{n\times n} \to \mathrm{Sym}(n) \cong \mathbb{R}^{n(n+1)/2}$, $A \mapsto A^T A$ and calculate its derivative.
c) Prove that $SL_n(\mathbb{R})$ is a submanifold of dimension $n^2 - 1$ and show that the tangent space of $SL_n(\mathbb{R})$ at the identity E is the space of traceless matrices, i.e., prove that $T_E SL_n(\mathbb{R}) = \{A \in M_n(\mathbb{R}) \colon \operatorname{tr}(A) = 0\}$.
Hint: Use the Taylor expansion $\det(E + tA) = 1 + t\operatorname{tr}(A) + O(t^2)$.

Problem 45 – Definitions:

Recall the following definitions:
- tangent vector
- vector field
- differentiable map and differential
- immersion and embedding (is an injective immersion an embedding?)
- submanifold

2. Vector fields, flows and the Frobenius theorem

Problem 46 – Vector fields on spheres:

a) Find a vector field on $\mathbb{S}^2$ which has only one zero.
b) On $\mathbb{S}^1$, the vector field $X(x_1, x_2) := (-x_2, x_1)$ has no zero. Find a vector field $X = X(x_1, \ldots, x_n)$ on $\mathbb{S}^n$ without zero.
Hint: $\mathbb{R}^{2n} = \mathbb{C}^n$ – what is the vector field X on $\mathbb{S}^1$ in complex notation?
c) On the submanifold $\mathbb{S}^2 \subset \mathbb{R}^3$, consider the vector field $X(x, y, z) := (-y, x, 0)$. Visualize X and prove it cannot be the gradient of a differentiable function $f \colon \mathbb{S}^2 \to \mathbb{R}$.
d) When would you expect a vector field on submanifold $M^n \subset \mathbb{R}^m$ to have a potential $f \colon M^n \to \mathbb{R}$?

Problem 47 – Vector fields and division algebras:

Assume that on some $\mathbb{R}^n$ there is the structure of a *division algebra*, that is, a bilinear map $\beta \colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, written as $(x, y) \mapsto xy$, such that all maps

$$\lambda_x \colon \mathbb{R}^n \to \mathbb{R}^n, \quad y \mapsto xy \qquad \text{and} \qquad \rho_y \colon \mathbb{R}^n \to \mathbb{R}^n, \quad x \mapsto xy$$

are bijective. We do not assume that the multiplication β is associative, but we assume there is a unit element $e \in \mathbb{R}^n$ with $ex = xe = x$ for all $x \in \mathbb{R}^n$. Prove the following:

a) If $n > 1$ and $x \notin \mathbb{R}e$ then λ_x has no real eigenvalues.
Hint: If $xy = \mu y$ then $(x - \mu e)y = 0$.

b) n is even. *Hint:* Recall a linear algebra result on eigenvalues.

c) We extend $b_n = e$ to a basis $(b_1, \dots, b_n)$ of $\mathbb{R}^n$ and consider the corresponding vector fields $X_j := X_{\lambda_{b_j}}$ for $j = 1, \dots, n$ on $\mathbb{S}^{n-1}$. Show that for each $x \in \mathbb{S}^{n-1}$, the vectors $X_1(x), \dots, X_{n-1}(x)$ are linearly independent.
Hint: $\operatorname{span}\{x, b_1 x, \dots, b_{n-1}x\} = \rho_x(\mathbb{R}^n) = \mathbb{R}^n$.

d) An n-manifold is *parallelizable* if there are n vector fields which give a basis of each tangent space. Show that $\mathbb{S}^{n-1}$ is parallelizable if $\mathbb{R}^n$ carries the structure of a division algebra.

e) Show that the matrix group

$$\mathbb{H} := \left\{ \begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix} : a, b \in \mathbb{C} \right\}$$

gives $\mathbb{R}^4 = \mathbb{C}^2$ the structure of a four-dimensional associative division algebra, called quaternions.

Problem 48 – Index of a vector field on a surface:

Suppose a vector field $X \in \mathcal{V}(\mathbb{R}^2)$ has only a discrete set Z of zeros. For any differentiable loop (closed curve) $c(t)$ in $\mathbb{R}^2 \setminus Z$, define the number

$$i(X, c) := \frac{1}{2\pi} \int \varphi'(t)\, dt, \qquad \text{where } \varphi(t) := \angle(X(c(t)), E) \text{ is continuous,}$$

as the total change of angle along c which X makes against a constant vector field $E \neq 0$.

a) Prove that $i(X, c)$ does not depend on E.

b) Prove that loops c_1, c_2 which are (differentiably) homotopic in $\mathbb{R}^2 \setminus Z$ have the same index, $i(X, c_1) = i(X, c_2)$.

c) Let $p \in Z$ and c be a loop in $\mathbb{R}^2 \setminus Z$ which is null homotopic in $\{p\} \cup (\mathbb{R}^2 \setminus Z)$ and has winding number $+1$ about p. Then the *index* $j(X, p)$ of X at p is defined by $j(X, p) := i(X, c)$. (Compare with the beautiful pictures on p. 109 of Hopf's book: Differential Geometry in the Large)

d) In case you know about Riemannian geometry: While here we use the angle with respect to the standard Riemannian metric $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^2$, we can also use any Riemannian metric g on $\mathbb{R}^2$. Prove that the similarly defined number $i(X, c) := i(g, X, c)$ agrees.

e) Extensions: Reason that $i(X, c)$ is defined on differentiable manifolds M. Do you have any idea for a similar number in higher dimensions?

Problem 49 – Preparation for Lie derivatives:

a) Let $f\colon \mathbb{R}^m \to \mathbb{R}^n$. Define the directional derivative of f at $p \in \mathbb{R}^m$ with respect to a direction $\xi \in \mathbb{R}^m$.
b) Relate the directional derivative to the differential; state the result also with sums and indices, avoiding matrix notation.

Problem 50 – Flows generated by vector fields:

Sketch the following vector fields on $\mathbb{R}^2$ and determine the flow for each them:

$$\text{a) } X\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}x\\y\end{pmatrix}, \qquad \text{b) } Y\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}x\\-y\end{pmatrix}, \qquad \text{c) } Z\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}x\\2y\end{pmatrix}$$

Problem 51 – Expansion of a flow:

For $X \in \mathcal{V}(\mathbb{R}^n)$ verify the expansion $\varphi_t(p) = p + tX(p) + O(t^2)$ at $t = 0$.

Problem 52 – Flows on compact manifolds:

Let M be manifold and $X \in \mathcal{V}(M)$ a vector field.

a) Suppose $c\colon I \to M$ is a maximal integral curve of X. Prove that in case $I \neq \mathbb{R}$ the manifold M does not have a compact subset $K \subset M$ containing $c(I) \subset K$.
b) Conclude that on a compact manifold, the maximal flow φ of X is global.

Problem 53 – Global flows:

a) Determine the flows of the following two vector fields on $\mathbb{R}^2$. Are they global?
$$X(x,y) := x^2 e_2, \qquad Y(x,y) = x^2 e_1,$$
b) Does every vector field on the real line $\mathbb{R}$ generate a global flow?
c) Prove that every compactly supported vector field X on a manifold M generates a global flow.

Problem 54 – Homogeneity:

Suppose M is a connected manifold and $p, q \in M$ where $p \neq q$. We want to show that there is a diffeomorphism $f\colon M \to M$ with $f(p) = q$.

a) Show that M is path-connected, that is, all $p, q \in M$ can be joined by a pieceweise differentiable injective curve $c\colon [0,1] \to M$ such that $c(0) = p$ and $c(1) = q$. Let us from now on assume that c is an embedding.
b) Extend the vector field $c'(t)$ along $c([0,1])$ to a compactly supported vector field X on all of M. *Hint:* Partition of unity.
c) Use the flow φ of X to construct the required diffeomorphism.

Problem 55 – Critical points and Hessian:

Let $f \in \mathcal{D}(M)$, $p \in M$ and (x, U) a chart of M at p. Then p is a *critical point of f* if $d(f \circ x^{-1})_{x(p)} = 0$.

a) Show that this definition is independent of the choice of chart.
b) Let p be a critical point of f. Prove
$$\partial_X \partial_Y f(p) = \partial_Y \partial_X f(p) \quad \text{for all vector fields } X, Y \in \mathcal{V}(M).$$
c) What is the least number of critical points on a compact manifold? Give an example. Speculate on the least number of critical points on a 2-torus; it might help to consider a height function for a torus of revolution in various positions.

Remark: Property b) allows to define a symmetric Hessian at a critical point p, and to assign an index to it, the so-called Morse index $\iota(p)$. A suitable sum over these indices equals the Euler characteristics of a manifold M and so is a tool to analyse the topology of M. The main application is for infinite-dimensional M, like the space of geodesics, etc.

Lie bracket

Problem 56 – Properties of the Lie bracket:

a) Prove $[fX, gY] = fg[X, Y] + f(\partial_X g)Y - g(\partial_Y f)X$ for all $f, g \in \mathbb{C}^\infty(M)$, $X, Y \in \mathcal{V}(M)$.
Hint: Calculate $\partial_{[fX,gY]} h$ for all $h \in \mathcal{D}(M)$.
b) Classify all one-dimensional and two-dimensional Lie algebras up to isomorphism.
Hint: For the two-dimensional case, show there is a basis $\{X, Y\}$ such that $[X, Y] = X$.
c) Verify that $[X, Y]$ is a derivation.
d) Verify the Jacobi identity for the Lie bracket.
Hint: Evaluate only one term and apply cyclic permutation.
e) Besides the cross product, there is another Lie bracket on $\mathbb{R}^3$, arsing from the so-called *Heisenberg group*: Let e_1, e_2, e_3 be the standard basis of $\mathbb{R}^3$ and define $[e_1, e_2] := e_3$ and $[e_i, e_j] := 0$ if $i = 3$ and j is arbitrary; continue this definition to all of $\mathbb{R}^3$ by anti-commutativity and linearity. Show that $(\mathbb{R}^3, [\cdot, \cdot])$ is a Lie-algebra.
f) Verify that the local representation of $[X, Y]$ transforms with the Jacobian of the transition map.

Problem 57 – Lie subalgebras:

a) A complex $n \times n$ matrix is *skew-Hermitian* if ${}^t\overline{A} = -A$. Prove that for each $n \in \mathbb{N}$, the set of skew-Hermitian matrices forms a Lie algebra, in particular, it is closed under $[A, B] = AB - BA$.
b) Find another such matrix algebra. *Hint:* trace

Problem 58 – Flows and Lie brackets:

Consider $X(u,v) := (0,u)$ on $\mathbb{R}^2$.

a) Plot $X(u,v)$.
b) Find a chart $(x,U)\colon U \to \mathbb{R}^2$ around the point $(1,0)$ such that $X = e_1$, as in Lemma 26. Formulate this first as a condition on the differential $dx\colon \mathbb{R}^2 \to \mathbb{R}^2$. What is the maximal choice of U?
c) If you like: Discuss all choices for x. Remember to verify that x is a diffeomorphism.
d) Moreover, let $Y(u,v) := (1,0)$, see the example in class. Verify Lemma 27 at the point $(1,0)$.

Problem 59 – Cylindrical coordinates:

On $\Omega := \mathbb{R}^3 \setminus \{(0,0,w) : w \in \mathbb{R}\}$ consider the vector fields $X(u,v,w) := \frac{1}{\sqrt{u^2+v^2}}(u,v,0)$ and $Y(u,v,w) := (J(u,v),0) = (-v,u,0)$.

a) Plot X and Y (the first two components!). Can you see what $[X,Y]$ is?
b) Verify they span an involutive distribution Δ.
c) Pick a point in Ω and determine a chart (x,U) as in Prop. 30.

Problem 60 – Non-integrable distribution:

Check explicitely that $X(p) = e_1$ and $Y(p) = e_2 + p^1 e_3$ is non-integrable.

Problem 61 – Lie bracket in the plane:

a) Give an example of two non-constant vector fields X, Y defined on the plane $\mathbb{R}^2$, such that the commutator vanishes, and another pair for which the commutator does not vanish.
b) For $X, Y \in \mathcal{V}(\mathbb{R}^2)$ the commutator is always of the form
$$[X,Y] = aX + bY \quad \text{where } a,b \in \mathcal{D}(\mathbb{R}^2).$$
Assume now that $[X,Y]_p \neq 0$ at $p \in \mathbb{R}^2$. Solve a differential equation to assert that there is a neighbourhood U of p together with functions
$$f,g \in \mathcal{D}(U) \quad \text{such that} \quad [fX, gY]_q \equiv 0 \quad \text{for all } q \in U.$$

Problem 62 – Heisenberg Algebra:

Besides the cross-product there is another non-trivial bracket which makes $\mathbb{R}^3$ into a Lie algebra. Consider the following vector fields:
$$X = e_1, \qquad Y(x) = e_2 + x_1 e_3, \qquad Z = e_3.$$

a) Sketch the three vector fields on the (x^1, x^2)-plane of $\mathbb{R}^3$. How do they extend to all of $\mathbb{R}^3$?
b) Calculate $[X, Y]$, $[X, Z]$ and $[Y, Z]$.
c) Consider $\{X, Y, Z\}$ as a basis of the vector fields on $\mathbb{R}^3$. Show that the bilinear and anti-commutative extension of $[\cdot, \cdot]$ onto $\mathrm{span}\{X, Y, Z\}$ on $\mathbb{R}^3$ defines a Lie algebra $\mathfrak{h} := (\mathbb{R}^3, [\cdot, \cdot])$.
d) Compute the flows φ of X and ψ of Y. What are their integral curves geometrically?
e) Consider a family of piecewise differentiable integral curves in $\mathbb{R}^3$ with initial point 0 and with x_3-projection a square of edgelength $r > 0$ in the (x_1, x_2)-plane, such that the four edges are tangent to $X, Y, -X, -Y$. Explain $[X, Y](0)$ geometrically, perhaps making use of your sketch.
f) Is there a surface, i.e., a submanifold $M \subset \mathbb{R}^3$ of dimension 2, such that its tangent space is spanned by X and Y?

Problem 63 – Heisenberg group:

Let $H \subset GL_n(\mathbb{R})$ be the Lie group (check!) of unit upper triangular matrices, i.e.

$$H := \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} : a, b, c \in \mathbb{R} \right\}$$

a) Is H abelian, i.e., commutative?
b) Compute the Lie algebra $\mathfrak{h}$ of H and show that the Lie bracket is closed.
c) Show that the Lie algebra can be identified with the one constructed in Problem 62.
d) Show that the elements of $\mathfrak{h}$ are nilpotent matrices.
e) Verify that the exponential map is a diffeomorphism from $\mathfrak{h}$ onto H.

Problem 64 – The Parking Theorem:

Consider the problem of parking a car in a free parking spot along the curb. We want to show that this can be done whenever the gap is longer than the length of the car, only using standard steering and forwards/backwards motion.

a) Suggest a reasonable configuration space Ω for the following four coordinates. The (x, y) coordinates of the center of the rear axis, the direction ϑ of the car, and the angle φ between the front wheels and the direction of the car. Which coordinates are sufficient to describe the position of the car?

The driver can only make the car move forwards or backwards, and steer. The following vector fields correspond to unit velocities in configuration space, and their integral curves can be realized:

$$\mathrm{Steer}(p) = (0, 0, 0, 1), \quad \mathrm{Drive}(p) = (\cos(\vartheta), \sin(\vartheta), \tfrac{\tan(\varphi)}{L}, 0) \quad \text{where } p = (x, y, \vartheta, \varphi) \in \Omega.$$

Here L is the length of the car, which for simplicity we assume to agree with the distance between front and rear axis.

b) Show that the Lie bracket satisfies

$$[\mathrm{Steer}(p), \mathrm{Drive}(p)] = f(p)\,\mathrm{Turn}(p)$$

where f is a differentiable non-vanishing function on Ω and $\mathrm{Turn}(x) := (0, 0, 1, 0)$. Compute f explicitly.

c) Explain how we can use Steer and Drive to obtain the same configuration change as an integral curve of Turn.

The parking spot can be shorter than the diagonal of the car. Therefore, it is not obvious that Steer, Drive, and Turn are sufficient to reach the parking position. However, the following commutator is useful:

d) Compute $\mathrm{Slide}(p) := [\mathrm{Turn}(p), \mathrm{Drive}(p)]$.

e) Show that Steer, Drive, Turn, and Slide form a basis of the vector fields $\mathcal{V}(\Omega)$.

f) Explain how Steer and Drive may be used to park the car, provided the parking spot is longer than the length of the car.

3. Differential forms and Stokes' theorem

Problem 65 – Skew symmetric bilinear forms:

The purpose of this problem is to prepare the class on multilinear algebra.

a) Give an example of a skew-symmetric bilinear form, $b\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, that is, $b(v, w) = -b(w, v)$ for all $v, w \in \mathbb{R}^n$.

b) Show that $b(v, v) = 0$ for all $v \in \mathbb{R}^n$ is equivalent to b skew-symmetric.

c) What is the dimension of the space of skew-symmetric forms $B(n)$? Exhibit a basis for $B(n)$, for instance in terms of the basis $e^i = \langle ., e_i\rangle$ of the dual space.

d) Can you find a projection which maps an arbitrary bilinear form to the skew-symmetric forms? What are reasonable properties to be demanded?

Problem 66 – Quiz multilinear algebra:

a) True or false for $\omega, \eta \in \Lambda^1 V$, $a \in \mathbb{R}$?:

$\square$ $a\omega \wedge a\eta = a(\omega \wedge \eta)$, $\square$ $a\omega \wedge \eta = \omega \wedge a\eta$, $\square$ $\omega \wedge \eta = \eta \wedge \omega$, $\square$ $(\omega - \eta) \wedge (\omega + \eta) = 2\omega \wedge \eta$.

Do you obtain the same answers for the case $\omega, \eta \in \Lambda^k V$?

b) Write $e^i \wedge e^j(e_k, e_l)$ in terms of Kronecker-δ's.

c) What is the dimension of $\Lambda^2\mathbb{R}^4$? Provide a basis.

d) Determine $\{v \in \mathbb{R}^2 : e^1 \wedge e^2(v, e_2) = 0\}$

e) For $w \in \mathbb{R}^3$ determine $V(w) := \{v \in \mathbb{R}^3 : e^1 \wedge e^2(v, w) = 0\}$
f) Determine $L := \{\omega \in \Lambda^2\mathbb{R}^n : \omega(e_1, e_2) = 0\}$ by using a basis representation for ω.
g) True or false: $(g \circ f)^*\omega = g^*(f^*(\omega))$ for $\omega \in \Lambda M$, $f, g\colon M \to M$.

Problem 67 – grad, curl, div *in* $\mathbb{R}^3$*:*

a) For $f \in C^\infty(\mathbb{R}^3, \mathbb{R})$, prove that

$$df = \sum_{i=1}^{3} \operatorname{grad}(f)_i e^i$$

where $\operatorname{grad} f = (\partial_1 f, \partial_2 f, \partial_3 f)$.

b) Let $g \in C^\infty(\mathbb{R}^3, \mathbb{R}^3)$ and $\omega = g_1 e^1 + g_2 e^2 + g_3 e^3$. Show that

$$d\omega = \operatorname{curl}(g)_1 e^2 \wedge e^3 + \operatorname{curl}(g)_2 e^3 \wedge e^1 + \operatorname{curl}(g)_3 e^1 \wedge e^2$$

where $\operatorname{curl}(g) = (\partial_2 g_3 - \partial_3 g_2, \partial_3 g_1 - \partial_1 g_3, \partial_1 g_2 - \partial_2 g_1)$.

c) Let $h \in C^\infty(\mathbb{R}^3, \mathbb{R}^3)$ and $\eta = h_1 e^2 \wedge e^3 + h_2 e^3 \wedge e^1 + h_1 e^1 \wedge e^2$. Prove that

$$d\eta = \operatorname{div}(h) e^1 \wedge e^2 \wedge e^3$$

where $\operatorname{div} h = \partial_1 h_1 + \partial_2 h_2 + \partial_3 h_3$.

d) Show that $\operatorname{div} \circ \operatorname{curl}$ and $\operatorname{curl} \circ \operatorname{grad} = 0$.

Problem 68 – Geometric interpretation of a two-form:

Let $P(v, w)$ be the planar parallelogram in $\mathbb{R}^3$ spanned by $v, w \in \mathbb{R}^3$. Let $\pi\colon \mathbb{R}^3 \to \mathbb{R}^2$ be projection to the xy-plane.

a) Give a formula for the signed area of $\pi(P(v, w))$.
b) Now consider $\eta := e^1 \wedge e^2 \in \Lambda^2\mathbb{R}^3$. Prove that $\eta(v, w)$ agrees with the signed area of $\pi(P(v, w))$.
c) What changes when we replace $\mathbb{R}^3$ by $\mathbb{R}^n$?

Problem 69 – Decomposable and indecomposable 2*-forms:*

a) Consider the forms $\omega := e^1 \wedge e^2$ and $\eta := e^1 \wedge e^3 \in \Lambda^2(\mathbb{R}^3)^*$. Determine $y \in (\mathbb{R}^3)^*$ such that $\omega + \eta = e^1 \wedge y$
b) Let $v, w \in V^*$ be linearly independent. For any nonzero $x \in \operatorname{span}\{v, w\}$ find $y \in \operatorname{span}\{v, w\}$ such that $v \wedge w = x \wedge y$.
c) Show that on $\mathbb{R}^3$ any two-forms $\omega := v \wedge w$ and $\eta := r \wedge s$ have a sum $\omega + \eta = x \wedge y$ for some covectors $x, y \in (\mathbb{R}^3)^*$.
d) Prove that $e^1 \wedge e^2 + e^3 \wedge e^4 \in \Lambda^2\mathbb{R}^4$ cannot be written in the form $x \wedge y$ for $x, y \in (\mathbb{R}^4)^*$.
e) Find $\omega \in \Lambda^2\mathbb{R}^4$ such that $\omega \wedge \omega \neq 0$.

Problem 70 – Decomposable k-forms:

Let $\omega \in \Lambda^k V$. We say ω is *decomposable*, if there exists $\omega^1, \dots, \omega^k \in \Lambda^1 V$ such that $\omega = \omega^1 \wedge \cdots \wedge \omega^k$.

a) Let $\omega \in \Lambda^k V$ be decomposable. Calculate $\omega \wedge \omega$.
b) Let $\dim V \geq 4$ and $\omega^1, \dots, \omega^4$ be linearly independent. Is $\omega^1 \wedge \omega^2 + \omega^3 \wedge \omega^4$ decomposable?
c) Prove that if $\dim V \leq 3$, then every $\omega \in \Lambda^k V$ is decomposable.
d) If $\dim V = 4$, give an example of a non-decomposable element $\omega \in \Lambda^k V$.

Problem 71 – Linear independence of one-forms:

Prove that differential 1-forms $\omega^1, \dots, \omega^k$ on a manifold M^n are linearly independent if and only if $\omega^1 \wedge \cdots \wedge \omega^k \neq 0$.

Problem 72 – 1-forms and vector fields:

Prove that

$$\Phi \colon \mathcal{V}(\mathbb{R}^n) \to \Lambda^1 \mathbb{R}^n, \qquad \Phi(X) = \langle X, \cdot \rangle$$

is bijective. Which linearity of Φ can you assert?

Problem 73 – Quiz differential:

a) Calculate the differential of $e^x \cos y \, dx - e^x \sin y \, dy$
b) What does the invariant formula for $d\omega$ give in case $\omega \in \Lambda^0 M$?
c) Tick those expressions which are $\mathcal{D}(M)$-linear in X:
   □ $[X, Y]$, □ $d\omega(X, Y, Z)$, □ $\partial_X(\omega(Y, Z))$ for $\omega \in \Lambda^2 M$.
d) For given $f \in \mathcal{V}(\mathbb{R}^n)$ find a form ω on $\mathbb{R}^n$ such that $d\omega = \operatorname{div} f \; e^1 \wedge \ldots \wedge e^n$.
e) Let $\omega \in \Lambda^k M$ and $X_1, \dots, X_k \in \mathcal{V}(M)$. Which of the following statements about the value of $d\omega(X_1, \dots, X_k)$ at $p \in M$ is true?
   - It only depends on the values of the X_i's at p, but not on the way they extend to M.
   - It only depends on the value of ω at p but not of the way the form ω extends to M.

Problem 74 – Hodge Star:

Consider the Euclidean vector space $\mathbb{R}^n, \langle \cdot, \cdot \rangle$. Let $(e_1, \dots, e_n)$ be an orthonormal basis, and $(e^1, \dots, e^n)$ be the dual basis. The *Hodge star operator* is a linear operator defined by its action on a basis, namely

$$* \colon \Lambda^k V \to \Lambda^{n-k} V, \qquad *(e^{i_1} \wedge e^{i_2} \wedge \cdots \wedge e^{i_k}) := e^{i_{k+1}} \wedge e^{i_{k+2}} \wedge \cdots \wedge e^{i_n};$$

here $\{i_1, \dots, i_k, i_{k+1}, \dots, i_n\}$ is an even permutation of $\{1, 2, \dots, n\}$; for an odd permutation we take the negative of the right hand side.

a) Compute $*(e^1 \wedge e^2)$ in $\mathbb{R}^3$ and $*1$ in $\mathbb{R}^n$. Find an eigenvector and eigenvalue of $*$ on $\Lambda^2\mathbb{R}^4$.
b) Show that $*$ is well-defined, i.e., independent of the permutation. What are the dimensions of $\Lambda^k\mathbb{R}^n$ and $\Lambda^{n-k}\mathbb{R}^n$, and why is $*$ a vector space isomorphism?
c) Prove $** = (-1)^{k(n-k)}$.
d) Prove that $\langle v, w\rangle := *(w \wedge *v)$ defines an inner product on $\Lambda^k\mathbb{R}^n$.

We use the Hodge star operator to define the *codifferential*

$$d^* := (-1)^{n(k+1)+1} * d*\colon \Lambda^k\mathbb{R}^n \to \Lambda^{k-1}\mathbb{R}^n.$$

e) Show that $(d^*)^2 = 0$.
f) Define the *Laplace-de Rham* operator by $\Delta := dd^* + d^*d$ and show that for $\mathbb{R}^3$ it coincides with the usual Laplacian $-\Delta f = \sum_{i=1}^3 \partial_i^2 f$.

Remark: On a manifold M, Hodge star and codifferential are defined once each tangent space T_pM is Euclidean, that is, if the manifold carries a Riemannian metric g. To verify this statement, it must only be shown that the definition of $*$ is independent of the choice of oriented orthonormal basis. In fact, only the non-degeneracy of the inner product is needed, and so our definitions still work on the Lorentz manifolds used in general relativity.

Problem 75 – Maxwell equations:

When formulated in the language of differential forms, the Maxwell equations of electrodynamics on space-time $\mathbb{R}^4$ attain the elegant form

$$dF = 0 \quad \text{and} \quad d^*F = j \qquad \text{for } F \in \Lambda^2\mathbb{R}^4.$$

Here the units are such that the speed of light is 1, the operators d and d^* are differential and codifferential, respectively, and the 1-form j is defined below.

According to the first equation the *electromagnetic field tensor* F is closed. The Poincaré-Lemma holds for $\mathbb{R}^4$, and so F is exact as well. That is, $F = dA$ for a 1-form $A \in \Lambda^1\mathbb{R}^4$ which is called the *electromagnetic vector potential.*

For the following we consider coordinates (x, y, z, t) of $\mathbb{R}^4$. With respect to the coordinate 1-forms (dx, dy, dz, dt), dual to the standard basis, F reads

$$F = E_1\, dx \wedge dt + E_2\, dy \wedge dt + E_3\, dz \wedge dt + B_1\, dy \wedge dz + B_2\, dz \wedge dx + B_3\, dx \wedge dy.$$

Here the coefficients are functions on space-time: $E = (E_1, E_2, E_3)$ is the *electric field* and $B = (B_1, B_2, B_3)$ the *magnetic field.*

a) Verify that the first Maxwell equation $dF = 0$ is equivalent to the Gauss's law for magnetism $\operatorname{div}(B) = 0$ where the divergence is with respect to the first three coordinates, and to the Faraday induction law $\operatorname{curl}(E) = -\partial_t B$ where ∂_t denotes ∂_4.
Hint: Use cyclic permutations for the computation, do not use alphabetic ordering.

b) In order to specify the second Maxwell equation we introduce the 1-form $j \in \Lambda^1 M$ by

$$j = i_1 dx + i_2 dy + i_3 dz - \rho\, dt,$$

where the coefficients are the charge density [*Ladungsdichte*] ρ and the current density [*elektrische Stromdichte*] $i = (i_1, i_2, i_3)$; in vacuum, j vanishes. Verify that $d^*F = j$ is equivalent to Gauss's law $\operatorname{div}(E) = \rho$ and Ampère's law $\operatorname{curl}(B) + \partial_t E = i$.

Remark: The first Maxwell equation $dF = 0$ can be formulated on any differentiable 4-manifold M. The equations for $\operatorname{div}(B)$ and $\operatorname{curl}(E)$ then become true for each choice of local coordinate (x, y, z, t), where div and curl have an invariant definition, assigned also through dF. However, for the codifferential to be defined, the second Maxwell equation $d^*F = 0$ requires a Riemannian metric on M, which is a pointwise inner product $g = g_p$ on T_pM. In fact, as for the Hodge-star, it suffices that g is only non-degenerate on each tangent space, and so the equation can be stated for the Lorentz-4-manifolds M of general relativity. We have avoided this extra complication, but have to pay the price that our version of Ampére's law assumes the physically incorrect sign for $\partial_t E$. While the Maxwell equations are not Galilei invariant (they include the absolute speed of light!) they can be shown to be invariant under Lorentz transformations, that is, under diffeomorphisms preserving g. This observation guided the development of general relativity. Let us also note that if M has topology the form F need not be exact, so that a vector potential A possibly exists only in a generalized sense.

Problem 76 – Symplectic vector space:

Let V be a vector space and let $\omega \in \Lambda^2 V$ be closed (i.e. $d\omega = 0$) non-degenerate (i.e. $\omega(w, v) = 0$ for all $v \in V$ implies $w = 0$). We call the pair (V, ω) a *symplectic vector space*. Furthermore, let W be a subspace of V. Define the *symplectic complement* by $W^\omega := \{v \in V : \omega(v, w) = 0 \text{ for all } w \in W\}$.

a) Let $p, q \in V$ be non-zero with $\omega(p, q) \neq 0$ and let $W := \operatorname{span}\{p, q\}$. Show that $V = W \oplus W^\omega$ with W and W^ω being symplectic.

b) Let (V, ω) a symplectic vector space. Show that there exists a basis $p_1, \dots p_n, q_1, \dots, q_n$ such that

$$\omega(p_j, p_k) = \omega(q_j, q_k) = 0 \qquad \text{and} \qquad \omega(p_j, q_k) = \delta_{jk}.$$

c) Conclude that a symplectic vector space is even-dimensional and show that it is orientable.

d) Let $H \in C^\infty(V)$ and let X_H be a vector field such that $\omega(X_H, \cdot) = dH$. Show that a curve $c(t) = (p_1(t), \dots, p_n(t), q_1(t), \dots, q_n(t))$ is an integral curve of X_H if and only if $\partial_t p_j = -\partial_{q_j} H$ and $\partial_t q_j = \partial_{p_j} H$ for all $j = 1, \dots, n$.
Remark: The equations $\partial_t p_j = -\partial_{q_j} H$ and $\partial_t q_j = \partial_{p_j} H$ are the *Hamilton equations* in classical mechanics.

Cubes and chains

Problem 77 – n-dimensional cube:

Denote the standard unit cube by $C := \{x \in \mathbb{R}^n : 0 \leq x^1, \ldots, x^n \leq 1\}$.

a) List the faces of C. How many are there?
b) For $1 \leq i \leq n$ let
$$\omega_i \in \Lambda^{n-1}\mathbb{R}^n, \qquad \omega_i := e^1 \wedge \ldots \wedge \widehat{e^i} \wedge \ldots \wedge e^n.$$
Describe those faces of C such that the form ω_i vanishes on multivectors formed by vectors tangent to the faces.

Problem 78 – Quiz:

True or false?:
□ The image of a singular 2-cube in $\mathbb{R}^2$ can be a circle.
□ A chain $\sigma = \sum_i a^i \sigma_i$ is closed if and only each σ_i is closed.
□ If σ is a k-cube in $\mathbb{R}^n$, and $\omega \in \Lambda^k\mathbb{R}^n$ then $\int_\sigma \omega = 0$ whenever $k < n$.
□ If $\varphi\colon [0,1] \to [0,1]$ is a diffeomorphism, c a curve in $\mathbb{R}^n$, and $X \in \mathcal{V}(\mathbb{R}^n)$ then $\int_{c\circ\varphi} X\,ds = \int_c X\,ds$ (path integrals).
□ A 1-cube σ cannot be closed.

Problem 79 – Integration example:

Let $\sigma\colon [0,1]^2 \to \mathbb{R}^3$, $\sigma(x,y) = (\cos(x), \sin(x), y)$ parameterize a piece of the cylinder. Calculate
$$\int_{[0,1]^2} \sigma^*(e^i \wedge e^j) \qquad \text{for } i,j \in \{1,2,3\},$$
and explain the results geometrically.

Problem 80 – Closed form which is not exact:

Let φ be a polar angle function, defined on a suitable domain $U \subset \mathbb{R}^2$, that is, $z \mapsto \varphi(z)$ is continuous with $z = |z|\, e^{i\varphi(z)}$ for $z \in U$.

a) Calculate $\eta := d\varphi$ and $d\eta = d^2\varphi$.
b) Let $P\colon (0,\infty) \times \mathbb{R} \to \mathbb{R}^2\backslash\{0\}$, $P(r,\vartheta) = (r\cos\vartheta, r\sin\vartheta)$ be polar coordinates. Show that $P^*\eta = d\vartheta$.

We define:
• A curve $c \in C^1([0,1], U)$ is *closed* if $c(0) = c(1)$, and
• c_0 is (differentiably) *homotopic* to c_1 if there exists $h \in C^0([0,1]^2, U)$ such that $t \mapsto h(s,t)$ is a closed differentiable curve for all $s \in [0,1]$, such that $h(0,t) = c_0$ and $h(1,t) = c_1$.

c) Prove that $\int_c \eta$ agrees for closed curves which are homotopic in $\mathbb{R}^2\backslash\{0\}$. Moreover, show that the closed curves $c_n(t) := e^{2\pi i\, nt}\colon [0,1] \to \mathbb{R}^2\backslash\{0\}$ are not homotopic for distinct values of $n \in \mathbb{Z}$.

d) Show η can be defined on $\mathbb{R}^2\backslash\{0\}$, and prove that there is no function $\varphi\colon \mathbb{R}^2\backslash\{0\} \to \mathbb{R}$ such that $\eta = d\varphi$.
Remark: In contrast, the *Poincaré Lemma* states that any form η with $d\eta = 0$ can be written as $\eta = d\omega$ for some ω, whenever the domain is contractible.

Problem 81 – Path integrals and exactness:

In physics and mathematics it is important to study how the path integral $\int_c \omega$ for a 1-form ω depends on the path c. Suppose $c^1, c^2\colon [0,1] \to \mathbb{R}^n$ are two curves, and consider the 1-chain $\sigma := c^1 - c^2$.

a) Prove that $\partial\sigma = 0$ if and only if the endpoints of c^1 agree with those of c^2. For this case, formulate the equality $\int_{c^1} \omega = \int_{c^2} \omega$ as a statement on σ, and exhibit a 2-chain τ such that $\partial\tau = \sigma$, that is, σ is exact.
b) Assume $\partial\sigma = 0$ and the claim of the Stokes theorem, $\int_\tau d\omega = \int_\sigma \omega$. State a condition for the path integral to depend only on the endpoints of the path. State the same condition for the representation $\omega = \langle X, \cdot\rangle$, where $X \in \mathcal{V}(\mathbb{R}^n)$.
c) Discuss which of the above steps remain valid when $\mathbb{R}^n$ is replaced by $\mathbb{R}^2 \setminus \{0\}$, say.

Problem 82 – Quiz: Stokes theorem for manifolds:

True or false?:
□ Stokes theorem also holds for 0-forms (functions) on $I := [0, \infty)$.
□ If M^n is compact orientable without boundary and $\omega \in \Lambda^{n-1}$ then $d\omega$ has a zero at some point $p \in M$.

Problem 83 – Green's formula:

Consider a closed domain $D \subset \mathbb{R}^2$ which is the image of a 2-cube σ, such that $\partial D = d\sigma$. Prove that the area $A(D)$ of D can be calculated as the boundary integral

$$A(D) = \int_{\partial D} \omega,$$

where $\omega := \frac{1}{2}(x\,dy + y\,dx)$. Be precise on the assumption on orientation you need to require.

Problem 84 – Cauchy Integral Theorem:

We can consider complex-valued differential forms $\omega\colon \mathbb{R}^n \times \cdots \times \mathbb{R}^n \to \mathbb{C}$ which are multilinear and alternating. Each such k-form can be decomposed in terms of $\omega := \operatorname{Re}\omega + i \operatorname{Im}\omega$ into real-valued forms $\operatorname{Re}\omega, \operatorname{Im}\omega \in \Lambda^k\mathbb{R}^n$. Moreover, we define differential and integral of ω over k-cubes σ by

$$d\omega := d(\operatorname{Re}\omega) + i\, d(\operatorname{Im}\omega), \qquad \int_\sigma \omega := \int_\sigma \operatorname{Re}\omega + i \int_\sigma \operatorname{Im}\omega$$

and we extend the exterior product by setting

$$\omega \wedge \eta := (\operatorname{Re}\omega \wedge \operatorname{Re}\eta - \operatorname{Im}\omega \wedge \operatorname{Im}\eta) + i(\operatorname{Re}\omega \wedge \operatorname{Im}\eta - \operatorname{Im}\omega \wedge \operatorname{Re}\eta).$$

This will guarantee that all calculations for forms remain valid in the complex setting, including Stokes theorem. On $\mathbb{R}^2 = \mathbb{C}$, let dx and dy denote the coordinate 1-forms. Applying d to the functions z and $\overline{z}$ then gives $dz = dx + i\,dy$ and $d\overline{z} = dx - i\,dy$.

a) Determine the basis ∂_z and $\partial_{\overline{z}}$, dual to dz and $d\overline{z}$, in terms of the standard basis vectors $\partial_x := e_1$ and $\partial_y := e_2$.
b) Calculate $dz \wedge d\overline{z}$.
c) Prove: A differentiable map $f\colon \mathbb{C} \to \mathbb{C}$ is holomorphic if and only if the complex 1-form $\omega := f\,dz$ is closed ($d\omega = 0$).
Hint: To calculate $d\omega$ write $f = u+iv$ and use u_x, u_y, v_x, v_y to denote partial derivatives.
d) As an immediate consequence, apply Stokes theorem for a cube to c). What do you get?
e) For $f\colon \mathbb{C} \to \mathbb{C}$ holomorphic consider the complex 1-forms

$$\eta := \frac{f(z)}{z - z_0}\,dz \qquad \text{and} \qquad \zeta := (\overline{z} - \overline{z}_0) f(z)\,dz,$$

as well as a 2-cube σ such that $\partial\sigma$ parameterizes $\partial B_\varepsilon(z_0)$ injectively and counter-clockwise. Verify

$$\int_{\partial\sigma} \eta = \frac{1}{\varepsilon^2} \int_{\partial\sigma} \zeta,$$

calculate the right hand side using Stokes theorem, and take the limit $\varepsilon \to 0$. Prove Cauchy's integral formula first for $\partial\sigma$. Moreover, state the same formula for τ homologous to σ, that is, if τ and σ differ by the boundary of a 2-cube.

Problem 85 – Winding Number and Fundamental Theorem of Algebra:

Let $\mathbb{C}^* := \mathbb{C} \setminus \{0\} = \mathbb{R}^2 \setminus \{0\}$ and consider for $n \in \mathbb{Z}$ and $R \in \mathbb{R}$ the loops

$$c_{R,n}\colon [0,1] \to \mathbb{C}^*, \qquad c_{R,n}(t) := Re^{2\pi n t}.$$

a) Compute $\int_{c_{R,n}} \eta$ where η is the angle differential from Problem 34 d).
b) Prove that there is a 2-cube $\sigma\colon [0,1]^2 \to \mathbb{C}^*$ with $c_{R,n} - c_{r,n} = \sigma$ where $0 < r < R$.
c) Let $f\colon \mathbb{C} \to \mathbb{C}$ be a polynomial, $f(z) = z^n + a_{n-1}z^{n-1} + \ldots + a_0$, of degree $n \geq 1$. Prove that for R large $f \circ c_{R,1} - c_{R,n}$ is the boundary of a chain τ in $\mathbb{C}^*$.
Hint: Write $f(z) = z^n(1 + \frac{1}{z}(\ldots))$ and use $c_{R^n,n} = (c_{R,1})^n$.
d) Prove that $f(z) = 0$ for some $z \in \mathbb{C}$.
Hint: If $f(z) \neq 0$ for all z with $|z| \leq R$ then $f \circ c_{R,1} - f \circ c_{0,1}$ is a boundary.

Problem 86 – Quiz:

Is the relation "orientation compatible" an equivalence relation on the set of charts?

Problem 87 – De Rham Cohomology and Poincaré Lemma:

Let $Z^k(M)$ be the set of all closed k-forms and $B^k(M)$ the set of all exact k-forms on a manifold M. As $d^2 = 0$, we have $B^k(M) \subset Z^k(M)$ i.e. all exact forms are closed. It is natural to ask when closed forms are exact. The Poincaré Lemma addresses this question. We define the *k-th de Rham cohomology group* of M as

$$H^k(M) = \begin{cases} Z^k(M)/B^k(M) & k \geq 1 \\ Z^k(M) & k = 0 \end{cases}$$

The de Rham cohomology group characterizes those closed forms that are not exact i.e. elements in any given coset are identical up to an exact form.

a) Determine $H^k(\{p\})$ for the 0-dimensional manifold $\{p\}$ and $k \in \mathbb{N}_0$.
b) Let $f\colon N \to M$ be a differentiable map between differentiable manifolds M, N. Show that the pullback f^* induces a map $H^k(f^*)\colon H^k(M) \to H^k(N)$.
c) Consider the embedding $i_t\colon N \to [0,1] \times N$ where $i_t(p) = (t,p)$. Show that

$$i_1^*\omega - i_0^*\omega = dh(\omega) + h(d\omega), \qquad \text{for all } \omega \in \Lambda^k([0,1] \times N),$$

where the action of $h(\omega) \in \Lambda^{k-1}N$ is as follows:

$$h(\omega)_p(X_1, \ldots, X_{k-1}) := \int_0^1 \omega_{(t,p)}(\partial_t, X_1, \ldots, X_{k-1})\, dt \text{ for } X_1, \ldots, X_{k-1} \in \mathcal{V}(N).$$

Hint: Choose coordinates. It suffices to consider either $\omega = \sum_{|I|=k} f(p,t)dx^I$ or $\omega = \sum_{|J|=k-1} f(p,t)dt \wedge dx^J$.

We say two maps $f, g\colon N \to M$ are *homotopic* if there exists a differentiable map $H\colon [0,1] \times N \to M$ such that $H(0,x) = f(x)$ and $H(1,x) = g(x)$ for all $x \in N$.

d) Prove that for f, g homotopic we have $H^k(f^*) = H^k(g^*)$.

We say two manifolds M, N are *homotopy equivalent* if there exist differentiable maps $f\colon M \to N$ and $g\colon N \to M$ such that $f \circ g$ and $g \circ f$ are homotopic to the identity map.

e) Show that if M and N are homotopy equivalent then $H^k(M) \cong H^k(N)$ for all k.
Hint: Consider the map $\overline{h} := h \circ H^*$.
f) Conclude the Poincaré Lemma: If M is contractible (i.e. homotopy equivalent to a point) then all closed differentiable k-forms are exact.