Manifolds

Exercise Sheet 7.

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Groupwork

Exercise G1 (True or False?)

True or False? Prove your answers.

a) For any $\alpha, \beta \in \Omega^\bullet(M, \mathbb{R})$, $\mathrm{d}(\alpha \wedge \beta) = \mathrm{d}\alpha \wedge \beta + \alpha \wedge \mathrm{d}\beta$.

b) Any exact form is closed.

c) $\mathrm{d} \circ \mathcal{L}_X = \mathcal{L}_X \circ \mathrm{d}$ (here X is a smooth vector field on a manifold M.)

d) Any diffeomorphism $F \colon U \to V$, where $U, V \subseteq \mathbb{R}^m$ are connected open sets, is orientation-preserving or orientation-reversing. What if U, V are connected orientable manifolds?

e) A volume form on a closed manifold is never exact. What about the noncompact case?

Exercise G2 (Green's theorem)

a) Carefully prove Green's theorem: Let $D \subseteq \mathbb{R}^2$ be a compact regular domain and let $P, Q \colon D \to \mathbb{R}$ be smooth functions. Then

$$\int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathrm{d}x \, \mathrm{d}y = \int_{\partial D} P \, \mathrm{d}x + Q \, \mathrm{d}y \, .$$

b) Let $D \subseteq \mathbb{R}^2$ be a compact regular domain whose boundary is a simple closed curve C. Show that the area of D is equal to $\mathrm{Area}(D) = \int_C x \, \mathrm{d}y = \int_C -y \, \mathrm{d}x$.

Application: show that the area enclosed by the ellipse with semi-axes a and b is equal to πab.

Exercise G3 (Divergence theorem)

Let X be a smooth vector field on $\mathbb{R}^3$ and let $D \subseteq \mathbb{R}^3$ be a compact regular domain.

Denote $\omega = \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$ the volume form of $\mathbb{R}^3$.

a) Show that

$$\int_{\partial D} i_X \omega = \int_D (\operatorname{div} X)\, \omega$$

where we have denoted $X = P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y} + R\frac{\partial}{\partial z}$ and $\operatorname{div} X := \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$.

b) Let N denote the unit outward-pointing vector to ∂D. By definition, the area element of ∂D is the 2 form $\sigma := (i_N \omega)_{|\partial D}$. Show that $i_X \omega = \langle X, N\rangle \sigma$.

Hint: write $X = \langle X, N\rangle N + Y$ where Y is tangent to ∂D, and show that $(i_Y \omega)_{|\partial D} = 0$.

c) Conclude the *divergence theorem*:

$$\int_D (\operatorname{div} X)\, \omega = \int_{\partial D} \langle X, N\rangle \sigma \ .$$

The right-hand side integral is called *flux of X along ∂D*.

d) Application: considering the vector field $X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}$, use the divergence theorem to check that the volume of the unit ball $B^3 \subseteq \mathbb{R}^3$ and the area of the unit sphere $S^2 \subseteq \mathbb{R}^3$ are related by $3\,\mathrm{Vol}(B^3) = \mathrm{Area}(S^2)$.

Exercise G4 (Non-orientability of the Klein bottle)

Let Γ be the group of transformations of $\mathbb{R}^2$ generated by $\tau\colon (x,y) \mapsto (x+1, y)$ and $\sigma\colon (x,y) \mapsto (1-x, y+1)$.

a) Show that Γ acts freely and properly discontinuously on $\mathbb{R}^2$. Show that $K^2 := \mathbb{R}^2/\Gamma$ is the Klein bottle and the projection $\pi\colon \mathbb{R}^2 \to K^2$ is a covering map.

b) Let ω be any 2-form on K^2. Denote $\tilde{\omega} := \pi^*\omega$. Show that $\sigma^*\tilde{\omega} = \tilde{\omega}$.

c) Show that $\tilde{\omega} = f\,\mathrm{d}x \wedge \mathrm{d}y$, where $f \in \mathcal{C}^\infty(\mathbb{R}^2, \mathbb{R})$ satisfies $f \circ \sigma = -f$.

d) Show that ω vanishes somewhere and conclude that K^2 is non-orientable.

Further Exercises: Hamiltonian mechanics

Newton's laws of motion describe the motion of a mechanical system in response to a force. In the 18th century Lagrange introduced the so-called action functional as the main tool to describe mechanical dynamics. Newton and Lagrange both formulated the theory of classical mechanics on the tangent bundle a the *configuration space*, which is a smooth finite-dimensional manifold M^m. In the 19th century Hamilton presented a formulation of classical mechanics on the cotangent bundle T^*M using symplectic geometry.

Remark: It is necessary to do Exercise F1 first, but Exercises F2, F3, F4 are independent.

Exercise F1 (Symplectic structure)

Let M be a smooth manifold. A *symplectic form* on M is a closed 2-form $\omega \in \Omega^2(M, \mathbb{R})$ such that ω is *nondegenerate*, i.e. $\omega_{|_p}$ is a nondegenerate bilinear form on T_pM for all $p \in M$.

a) Denote $(x^1, y^1, \ldots, x^m, y^m)$ the standard coordinates on $\mathbb{C}^m = \mathbb{R}^{2m}$. Prove that $\omega = \sum_{i=1}^m \mathrm{d}x^i \wedge \mathrm{d}y^i$ is a symplectic form on $\mathbb{C}^m$.

b) Let (M, ω) be a symplectic manifold. Show that $\dim M$ is even.

c) Let (M, ω) be a symplectic manifold with $\dim M := 2m$. Denote by $\omega^m := \omega \wedge \cdots \wedge \omega$. Show that ω^m is a volume form on M. Conclude that any symplectic manifold is orientable.

d) Local coordinates $(x^1, y^1, \ldots, x^m, y^m)$ such that $\omega = \sum \mathrm{d}x^i \wedge \mathrm{d}y^i$ are called *Darboux coordinates*.

   Can you describe a symplectic structure and Darboux coordinates on the torus T^2?

   NB: The Darboux theorem says that any symplectic manifold locally admits Darboux coordinates.

e) Let (M, ω) be a symplectic manifold and let $f \in \mathcal{C}^\infty(M, \mathbb{R})$. The *Hamiltonian vector field* or *symplectic gradient* of f is the vector field X_f defined by $i_{X_f}\omega = \mathrm{d}f$. Show that this definition is legit. If (x^i, y^i) are Darboux coordinates, find the Hamiltonians of the functions x^i and y^i.

Exercise F2 (Canonical symplectic structure in the cotangent bundle)

Let M be a smooth manifold and let $N = \mathrm{T}^* M$ be the total space of the cotangent bundle of M. Denote by $\pi \colon N \to M$ the canonical projection.

a) Let $(q^1, \ldots, q^m)$ be local coordinates on $U \subseteq M$. Show that any $\alpha \in \mathrm{T}^* U$ can be written $\alpha = \sum_{i=1}^m p_i \, \mathrm{d}q^i$. Show that $(q^1, \ldots, q^m, p_1, \ldots, p_m)$ is a system of local coordinates on $\mathrm{T}^* U \subseteq N$.

b) Let $\alpha \in N$. By definition, $\alpha \in \mathrm{T}_p^* M$ for some $p \in M$. For any $u \in \mathrm{T}_\alpha N$, put $\vartheta_{|_\alpha}(u) = \alpha(\mathrm{d}\pi(u))$. Show that ϑ is a smooth 1-form on N and that in the coordinates above $\vartheta = \sum_{i=1}^m p_i \, \mathrm{d}q^i$.

   ϑ is called the Liouville 1-form on $\mathrm{T}^* M$.

c) Show that $\omega = \mathrm{d}\vartheta$ is a symplectic form on N, and that (p_i, q^i) are Darboux coordinates.

ω is called the canonical symplectic structure on T^*M.

d) Let $\alpha \in \Omega^1(N, \mathbb{R})$. Show that $\alpha^*\vartheta = \alpha$ (*self-reproducing property*) and $\alpha^*\omega = \mathrm{d}\alpha$.

Exercise F3 (Hamilton equations and Liouville's theorem)

A *Hamiltonian system* is a triple (N, ω, H), where (N, ω) is a symplectic manifold and $H \in \mathcal{C}^\infty(N, \mathbb{R})$.

a) (Hamilton equations) Let $(p_1, q_1, \ldots, p_m, q_m)$ be Darboux coordinates on N.

Show that a curve $c\colon I \to N$ with coordinate representation $c(t) = (p_1(t), \ldots, q_m(t))$ is an integral curve of the Hamiltonian vector field X_H iff the following set of PDEs is fulfilled:

$$\partial_t p_i = -\frac{\partial H}{\partial q_i}, \qquad \partial_t q_i = \frac{\partial H}{\partial p_i}, \qquad i = 1, ..., m.$$

b) Show that the Lie derivative $\mathcal{L}_{X_H}(\omega)$ vanishes. Let $\varphi_t : T^*N \to T^*N$ be the flow of the vector field X_H. Show that φ_t preserves the symplectic form, i.e., $\Phi_t^*\omega = \omega$.

c) Let $A \subseteq N$. The *symplectic volume* of A is defined by $\mathrm{vol}(A) := \frac{1}{m!}\int_A \omega^m$. Show that φ_t preserves the symplectic volume, i.e. $\mathrm{vol}(\varphi_t(A)) = \mathrm{vol}(A)$ for all $A \subseteq N$.

Exercise F4 (Poisson bracket)

Let (N, ω) be a symplectic manifold. For $f, g \in \mathcal{C}^\infty(N, \mathbb{R})$ we define the *Poisson bracket*

$$\{f, g\} := \omega(X_f, X_g)\,.$$

a) Show that the ring $\mathcal{C}^\infty(N, \mathbb{R})$ together with the Poisson bracket is a Lie algebra.

b) Verify that in Darboux coordinates $(p_1, \ldots, q_m)$ we have

$$\{f, g\} = \sum_{i=1}^{m} \left(\frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} \right).$$

Let $H \in \mathcal{C}^\infty(N, \mathbb{R})$ (the "Hamiltonian function"). A function $f \in \mathcal{C}^\infty(N, \mathbb{R})$ is called a *first integral* of the Hamiltonian function H if it is constant along all integral curves of X_H.

c) Show that f is a first integral of H if and only if $\{f, H\} = 0$.

Further Exercise: The hairy ball theorem

The goal of the next exercise is to show the *hairy ball theorem*: any smooth vector field on an even-dimensional sphere must vanish somewhere.

The proof we propose is taken from [Lafontaine] and is based on a famous proof of Milnor.

Exercise F5 (Hairy ball theorem)

Let m be an even integer. We denote $S^m(r) \subseteq \mathbb{R}^{m+1}$ the sphere centered at the origin of radius $r > 0$ in $\mathbb{R}^{m+1}$. By contradiction, let us assume that X is a nowhere vanishing vector field on $S^m := S^m(1)$.

a) One can assume that $\|X\| = 1$ everywhere: why?

   Let $f_\varepsilon \colon S^m \to \mathbb{R}^{m+1}$ be defined by $f_\varepsilon(x) = x + \varepsilon X_x$. Show that if $\varepsilon > 0$ is small enough, then f_ε is a diffeomorphism from S^m to $S^m(\sqrt{1+\varepsilon^2})$.

b) Let $\omega = \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^{m+1}$ denote the volume element of $\mathbb{R}^{m+1}$, and let σ_r denote the area element of $S^m(r)$, defined by $\sigma_r := (i_N\omega)_{|S^m(r)}$, where N denotes the unit outward-pointing normal vector to $S^m(r)$. Let $A(r) := \int_{S^m(r)} \sigma_r$. Argue that $A(r) = r^m A(1)$.

c) Compute σ_r. (*) For $\varepsilon > 0$, let $r = \sqrt{1+\varepsilon^2}$ and argue that $f_\varepsilon^*\sigma_r$ is a volume form on S^m that depends polynomially on ε. Conclude that $B(\varepsilon) := \int_{S^m} f_\varepsilon^*\sigma_r$ is a polynomial function of ε.

d) Show that $A(r) = B(\varepsilon)$ where $r = \sqrt{1+\varepsilon^2}$ and conclude.