Manifolds
Exercise Sheet 6.

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Groupwork

Exercise G1 (True or false?)

Prove or disprove:

a) For any $\alpha \in \Omega^k(M, \mathbb{R})$ and $\beta \in \Omega^l(M, \mathbb{R})$,
$$\beta \wedge \alpha = (-1)^{kl} \alpha \wedge \beta \ . \tag{1}$$

b) $\alpha \wedge \alpha = 0$ for any $\alpha \in \Omega^k(M, \mathbb{R})$.

Warning! There was a mistake about this in the lecture.

c) Let $(x^1, \dots, x^m)$ be a system of coordinates on $U \subseteq M$.

Then $\{\mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k} \ , \ 1 \leqslant i_1 < \cdots < i_k \leqslant m\}$ is a basis of $\Omega^k(U, \mathbb{R})$.

d) For any $f \in \mathcal{C}^\infty(M, \mathbb{R})$ and $X \in \Gamma(\mathrm{T}\, M)$,
$$i_X(\mathrm{d}f) = \mathrm{d}f(X) = X(f) = \mathcal{L}_X f \ . \tag{2}$$

Exercise G2 (Computations in coordinates)

Consider $\alpha = x\,\mathrm{d}x - y\,\mathrm{d}z$ and $\beta = \mathrm{d}x \wedge \mathrm{d}y - x\,\mathrm{d}y \wedge \mathrm{d}z$ on $M = \mathbb{R}^3$.

a) Compute $\alpha \wedge \beta$ and $\beta \wedge \alpha$.

b) Compute $f^*\alpha$ and $f^*\beta$ where $f \colon M \to M$ is given by $(x, y, z) = (y, x, xyz)$.

c) Compute $\mathcal{L}_X \alpha$ and $\mathcal{L}_X \beta$ where $X = \frac{\partial}{\partial x}$.

d) Compute $i_X \alpha$ and $i_X \beta$ where $X = \frac{\partial}{\partial x}$.

e) Compute $\mathrm{d}\alpha$ and $\mathrm{d}\beta$.

Exercise G3 (A 2-form on $\mathbb{R}^3$)

Consider the 2-form on $\mathbb{R}^3$ given by

$$\omega = x\,\mathrm{d}y \wedge \mathrm{d}z + y\,\mathrm{d}z \wedge \mathrm{d}z + z\,\mathrm{d}x \wedge \mathrm{d}y\ . \tag{3}$$

a) Compute ω in spherical coordinates $(\rho, \varphi, \vartheta)$.

   We recall that these satisfy $(x, y, z) = (\rho \sin\varphi \cos\vartheta, \rho \sin\varphi \sin\vartheta, \rho \cos\varphi)$.

b) Compute $\mathrm{d}\omega$ both in Cartesian and in spherical coordinates, and verify that the two expressions you found are indeed the same 3-form.

c) Let $f\colon S^2 \to \mathbb{R}^3$ denote the inclusion of the unit sphere. Compute $f^*\omega$ in spherical coordinates. Show that $f^*\omega$ never vanishes.

Homework

To solve these exercises you need the notion of the exterior derivative which will be introduced in the next lecture.

Exercise H1 (Hodge Star) **8 points**

Consider $M = \mathbb{R}^n$. The *Hodge star operator* is a $C^\infty(\mathbb{R}^n)$-linear operator defined by

$$*\colon \Omega^k(\mathbb{R}^n) \to \Omega^{n-k}(\mathbb{R}^n), \qquad *(dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_k}) := dx^{i_{k+1}} \wedge dx^{i_{k+2}} \wedge \cdots \wedge dx^{i_n};$$

where $(i_1, \ldots, i_k, i_{k+1}, \ldots, i_n)$ is an even permutation of $\{1, 2, \ldots, n\}$; for an odd permutation we take the negative of the right hand side.

a) Show that $*$ is well-defined, i.e., independent of the permutation.

b) Compute $*(dx^1 \wedge dx^2)$ in $\mathbb{R}^3$ and $*1$ in $\mathbb{R}^n$. We call $\omega := *1$ *volume form*.

c) Prove $** = (-1)^{k(n-k)}\,\mathrm{id}$.

We use the Hodge star operator to define the *codifferential*

$$d^* := (-1)^{n(k+1)+1} * d*\colon \Omega^k(\mathbb{R}^n) \to \Omega^{k-1}(\mathbb{R}^n).$$

d) Show that $(d^*)^2 = 0$.

Remark: On a manifold M, Hodge star and codifferential are defined once each tangent space T_pM is Euclidean, that is, if the manifold carries a Riemannian metric g. To verify this statement, it must only be shown that the definition of $*$ is independent of the choice of oriented orthonormal basis. In fact, only the non-degeneracy of the inner product is needed, and so our definitions still work on the Lorentz manifolds used in general relativity.

Exercise H2 (A commutative diagram) **8 points**

Recall the classical vector calculus operators on $\mathbb{R}^n$: the gradient of a function $f \in C^\infty(\mathbb{R}^n)$ and the divergence of a vector field $X \in \Gamma(T\mathbb{R}^n)$ are defined by

$$\operatorname{grad} f = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i} \frac{\partial}{\partial x^i}, \qquad \operatorname{div} X = \sum_{i=1}^{n} \frac{\partial X^i}{\partial x^i}.$$

In addition, for $n = 3$ the curl of a vector field is defined by

$$\operatorname{curl} X = \left(\frac{\partial X^3}{\partial x^2} - \frac{\partial X^2}{\partial x^3} \right) \frac{\partial}{\partial x^1} + \left(\frac{\partial X^1}{\partial x^3} - \frac{\partial X^3}{\partial x^1} \right) \frac{\partial}{\partial x^2} + \left(\frac{\partial X^2}{\partial x^1} - \frac{\partial X^1}{\partial x^2} \right) \frac{\partial}{\partial x^3}.$$

The Euclidean metric on $\mathbb{R}^3$ yields an index-lowering isomorphism $\flat : \Gamma(T\mathbb{R}^3) \to \Omega^1(\mathbb{R}^3)$ by identifying $\partial/\partial x^i$ with dx^i. The interior product yields another map $\beta : \Gamma(T\mathbb{R}^3) \to \Omega^2(\mathbb{R}^3), X \mapsto i_X\omega$, where ω denotes the volume form on $\mathbb{R}^3$ (see H1b).

The relationships among all of these operators are summarized in the following diagram:

$$\begin{array}{ccccccc}
C^\infty(\mathbb{R}^3) & \xrightarrow{\operatorname{grad}} & \Gamma(T\mathbb{R}^3) & \xrightarrow{\operatorname{curl}} & \Gamma(T\mathbb{R}^3) & \xrightarrow{\operatorname{div}} & C^\infty(\mathbb{R}^3) \\
\downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \flat} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle *} \\
\Omega^0(\mathbb{R}^3) & \xrightarrow{d} & \Omega^1(\mathbb{R}^3) & \xrightarrow{d} & \Omega^2(\mathbb{R}^3) & \xrightarrow{d} & \Omega^3(\mathbb{R}^3)
\end{array}$$

a) Prove that the diagram commutes.

b) Conclude that $\operatorname{curl} \circ \operatorname{grad} \equiv 0$ and $\operatorname{div} \circ \operatorname{curl} \equiv 0$ on $\mathbb{R}^3$.

Exercise H3 (Maxwell equations) **8 points**

For the following we consider $\mathbb{R}^4$ with coordinates (t, x, y, z). We define vector fields

$$E := E_1 \frac{\partial}{\partial x} + E_2 \frac{\partial}{\partial y} + E_3 \frac{\partial}{\partial z}, \qquad B := B_1 \frac{\partial}{\partial x} + B_2 \frac{\partial}{\partial y} + B_3 \frac{\partial}{\partial z}$$

with coefficient functions $E_i, B_i \in C^\infty(\mathbb{R}^4)$. E is called *electric field* and B *magnetic field*. Further we define the *electromagnetic field tensor* $F \in \Omega^2(\mathbb{R}^4)$ by

$$F := (E^\flat \wedge dt) - *(B^\flat \wedge dt).$$

a) Verify that

$$F = E_1\, dx \wedge dt + E_2\, dy \wedge dt + E_3\, dz \wedge dt + B_1\, dy \wedge dz + B_2\, dz \wedge dx + B_3\, dx \wedge dy.$$

b) Show that $dF = 0$ is equivalent to the Gauss's law for magnetism $\operatorname{div}(B) = 0$ where the divergence is with respect to the first three last coordinates (x, y, z), and to the Faraday induction law $\operatorname{curl}(E) = -\partial_t B$.

c) We introduce the 1-form $j \in \Omega^1(\mathbb{R}^4)$ by

$$j = i_1 dx + i_2 dy + i_3 dz - \rho\, dt,$$

where the coefficients are the *charge density* ρ and the *current density* $i = (i_1, i_2, i_3)$. Show that $d^*F = j$ is equivalent to Gauss's law $\operatorname{div}(E) = \rho$ and Ampère's law $\operatorname{curl}(B) + \partial_t E = i$.

When formulated in the language of differential forms, the Maxwell equations of electrodynamics on spacetime $\mathbb{R}^4$ attain the elegant form

$$dF = 0 \quad \text{and} \quad d^*F = j \qquad \text{for } F \in \Omega^2(\mathbb{R}^4), j \in \Omega^1(\mathbb{R}^4).$$

Remark 1: According to the first equation the electromagnetic field tensor F is closed. The Poincaré-Lemma holds for $\mathbb{R}^4$, and so F is exact as well. That is, $F = dA$ for a 1-form $A \in \Omega^1(\mathbb{R}^4)$ which is called the *electromagnetic vector potential*.

Remark 2: The first Maxwell equation $dF = 0$ can be formulated on any differentiable 4-manifold M. The equations for $\operatorname{div}(B)$ and $\operatorname{curl}(E)$ then become true for each choice of local coordinates (t, x, y, z), where div and curl have an invariant definition, assigned also through dF. However, for the codifferential to be defined, the second Maxwell equation $d^*F = 0$ requires a Riemannian metric on M, which is a pointwise inner product $g = g_p$ on T_pM. In fact, as for the Hodge-star, it suffices that g is only non-degenerate on each tangent space, and so the equation can be stated for the Lorentz-4-manifolds M of general relativity. We have avoided this extra complication, but have to pay the price that our version of Ampére's law assumes the physically incorrect sign for $\partial_t E$. While the Maxwell equations are not Galilei invariant (they include the absolute speed of light!) they can be shown to be invariant under Lorentz transformations, that is, under diffeomorphisms preserving g. This observation guided the development of general relativity. Let us also note that if M has topology the form F need not be exact, so that a vector potential A possibly exists only in a generalized sense.