Manifolds
Exercise Sheet 4.

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Groupwork

Exercise G1 (Holomorphic maps)

Let $f\colon U \subseteq \mathbb{C} \to \mathbb{C}$ be a holomorphic function. Show that f is a local diffeomorphism if and only if f' does not vanish.

Exercise G2 (Dense curve on the torus)

Let $T \coloneqq S^1 \times S^1 \subseteq \mathbb{C}^2$ denote the torus, and let $\alpha \in \mathbb{R} - \mathbb{Q}$ be an irrational number. Consider the curve $\gamma : \mathbb{R} \to T$ given by

$$\gamma(t) = \left(e^{2\pi i t}, e^{2\pi i \alpha t}\right).$$

a) Show that γ is an injective immersion.

b) Show or admit that $\{e^{2\pi i \alpha n}, n \in \mathbb{Z}\}$ is dense in S^1.

   Hint: Recall that any subgroup G of $(\mathbb{R}, +)$ is either dense in $\mathbb{R}$ or of the form $a\mathbb{Z}$ (for some $a \geqslant 0$). Accepting or proving this fact, show that $G = \{m + \alpha n, (m,n) \in \mathbb{Z}^2\}$ is a dense subgroup of $\mathbb{R}$.

c) Derive from (b) that the subset $\gamma(\mathbb{Z})$ has a limit point (in fact, all of its elements are limit points). Conclude that γ is not an embedding.

d) Does the fact that γ is not an embedding automatically imply that $\gamma(\mathbb{R})$ is not an embedded submanifold?

e) Derive from (b) that $\gamma(\mathbb{R})$ is dense in T. Conclude that $\gamma(\mathbb{R})$ is not an embedded submanifold.

f) What if $\alpha \in \mathbb{Q}$?

Exercise G3 (Fiber bundle vs submersion)

a) Show that any smooth fiber bundle $\pi\colon E \to B$ is a submersion.

b) Is the converse true?

Exercise G4 (A level set)

Consider the map $F\colon \mathbb{R}^4 \to \mathbb{R}^2$ defined by $F(x, y, s, t) = (x^2 + y, x^2 + y^2 + s^2 + t^2 + y)$.

Show that $(0, 1)$ is a regular value of F, and that the level set $F^{-1}(0, 1)$ is diffeomorphic to S^2.

Homework

Hand in your work by Tuesday, 16.06.2020.

Exercise H1 (Properties of local diffeomorphisms) **12 points**

True or False? Prove it.

a) Any composition of local diffeomorphisms is a local diffeomorphism.

b) Any local diffeomorphism is an open map / closed map.

c) Every diffeomorphism is a bijective local diffeomorphism, and the converse is also true.

d) A smooth map between manifolds of the same dimension is a local diffeomorphism iff it is an immersion iff it is a submersion.

Hints for solution:

a) The map $f : M \to N$ is a local diffeomorphism iff df_p is invertible for all $p \in M$ (see lecture). Since the composition of invertible maps is invertible, the statement follows.

b) Any local diffeomorphism is a local homeomorphism and thus an open map. In general, it is not a closed map: Let $U \subset M$ be an open subset of N that is not closed. Then the inclusion $f : U \to N$ is a local diffeomorphism, but not closed as U is closed in U, but $f(U) = U$ is not closed in N.

c) Follows directly from the definition.

d) Since df_p is linear, the statement follows from linear algebra.

Exercise H2 (Veronese Embedding) **13 points**

Define the map

$$f\colon S^2 \to \mathbb{R}^6, \quad (x,y,z) \mapsto (x^2, y^2, z^2, \sqrt{2}\,xy, \sqrt{2}\,yz, \sqrt{2}\,zx).$$

a) Prove that f is an immersion. Is it an embedding?

b) Show that $f(S^2)$ is contained in $S^5 \subset \mathbb{R}^6$ and in a hyperplane of $\mathbb{R}^6$. Conclude that f can be assumed to take an image in S^4_r for some $0 < r < 1$.

c) Prove that $f(S^2)$ is a proper subset of S^4_r using Sard's theorem. Compose f with a suitable map to find an immersion $F\colon S^2 \to \mathbb{R}^4$.

d) Show that F induces an embedding of $\mathbb{R}P^2$ into $\mathbb{R}^4$, the *Veronese embedding*.
Hint 1: Use H1 d) of the second sheet.
Hint 2: Prove or accept that if $f\colon M \to N$ is an injective immersion and M compact, then f is an embedding.

Hints for solution:

a) We have

$$df = \begin{pmatrix} 2x & 0 & 0 & \sqrt{2}y & 0 & \sqrt{2}z \\ 0 & 2y & 0 & \sqrt{2}x & \sqrt{2}z & 0 \\ 0 & 0 & 2z & 0 & \sqrt{2}y & \sqrt{2}x \end{pmatrix}^T .$$

Since $x^2+y^2+z^2 = 1$, we have that the matrix has full rank for all $x,y,z \in S^2$. Hence, df is injective and f is an immersion. Since $f(1,0,0) = f(-1,0,0)$, f is not injective and thus not an embedding.

b) We have

$$\begin{aligned} |f(x,y,z)|^2 &= (x^2)^2 + (y^2)^2 + (z^2)^2 + (\sqrt{2}xy)^2 + (\sqrt{2}yz)^2 + (\sqrt{2}zx)^2 \\ &= x^4 + y^4 + z^4 + 2x^2y^2 + 2y^2z^2 + 2z^2x^2 = (x^2+y^2+z^2)^2 \\ &= |(x,y,z)|^4 = 1. \end{aligned}$$

Hence, $f(S^2)$ is contained in $S^5 \subset \mathbb{R}^6$.

On the other hand we have $f_1 + f_2 + f_3 = 1$. That defines a hyperplane $H^5 \subset \mathbb{R}^6$ where f_4, f_5, f_6 are arbitrary. Note that this hyperplane intersects $\mathbb{S}^5$: It contains, for instance, the pole $p_1 = (1,0,0,0,0,0) = f(1,0,0)$, but H is not equal to the hyperplane $p_1^\perp = f_1 = 0$. We have $H = (1,1,1,0,0,0)^\perp$. Thus the two hyperplanes are transversal: $\langle p_1, (1,1,1,0,0,0)/\sqrt{3}\rangle 1/\sqrt{3} = \cos\alpha$ with $\alpha \approx 54°$. Therefore $H \cap S^5$ is neither empty, nor a point, and so must be an S^4_r.

c) Since $\dim(S^2) = 2 < 4 = \dim(S^4_r)$, by Sard's theorem $f(S^2)$ is negligible in S^4_r. In particular, it is a proper subset and we can compose f with the stereographic projection (maybe after some suitable rotation) to obtain a map $F : S^2 \to \mathbb{R}^4$. Since f is an immersion and the stereographic projection a diffeomorphism, F is an immersion.

d) Let (x, y, z) $(x', y', z') \in S^2$ with $f((x, y, z)) = f((x', y', z'))$. Comparing them yield $(x, y, z) = (\pm x', \pm y', \pm z')$, which are equivalent for F. Hence, F is an injective immersion. Since $\mathbb{R}P^2$ is compact, F is an embedding.

Exercise H3 (Orthogonal group) **5 points**

Prove carefully that $\mathsf{O}(n, \mathbb{R})$ is a matrix Lie group.

Hints for solution:
Let $\mathsf{Sym}(n, \mathbb{R})$ denote the space of symmetric $n \times n$-matrices over $\mathbb{R}$. Consider the map

$$f : \mathsf{M}(n, \mathbb{R}) \to \mathsf{Sym}(n, \mathbb{R}), \quad A \mapsto A^T A.$$

Then $\mathsf{O}(n, \mathbb{R}) = f^{-1}(E)$. The differential $df_E = A^T + A$ is surjective, because for $B \in \mathsf{Sym}(n, \mathbb{R})$ and $C \in \mathsf{O}(n, \mathbb{R})$ it holds $df_E(\frac{1}{2}BC) = B$. Thus, E is a regular value and $\mathsf{O}(n, \mathbb{R})$ is a submanifold of $\mathsf{GL}(n, \mathbb{R})$. Since $\mathsf{O}(n, \mathbb{R})$ is also a subgroup of $\mathsf{GL}(n, \mathbb{R})$ and $\mathsf{GL}(n, \mathbb{R})$ is itself a Lie group, the statement follows.

Further Exercises

Exercise F1 (Proper maps, immersions and embeddings)

Let X and Y be locally compact Hausdorff topological spaces. A map $f : X \to Y$ is called *proper* if it is continuous and the preimage of any compact subset of Y is a compact subset of X.

a) Show that a proper map is closed (the image of any closed set is closed).

b) Derive that an injective proper map is a topological embedding.

c) Consider a smooth map $f : M \to N$. Show that if f is an injective proper immersion, then f is a smooth embedding. Is the converse true?

d) Show that an embedding $f : M \to N$ is proper iff $f(M)$ is a closed subset of N.

Exercise F2 (*) (Easy Sard theorem)

Prove the easy case of Sard's theorem: if $f : M \to N$ is a smooth map, and $\dim M < \dim N$, then the image of f is a negligible subset of N. Is this still true if f is only assumed continuous?

Exercise F3 (Characterizations of submanifolds)

Prove the theorem of Chap. 6 characterizing submanifolds of $\mathbb{R}^n$ (see the lecture PDF for a precise statement, and elements of proof).

Exercise F4 (Determinant and special linear group)

a) Prove that $\det\colon \mathrm{M}(n,\mathbb{R})$ is a smooth map. Compute its differential at $M = I_n$, then at any $M \in \mathrm{M}(n,\mathbb{R})$.

b) Prove that $\det$ is a submersion on $\mathsf{GL}(n,\mathbb{R})$.

c) Conclude that $\mathrm{SL}(n,\mathbb{R})$ is a matrix Lie group.

Exercise F5 (Easy Whitney theorem)

Prove that the "easiest Whitney theorem" implies the "easy Whitney theorem", in other words:

Let M be a smooth manifold of dim. m that admits an embedding to $\mathbb{R}^N$ for some $N \in \mathbb{N}$. Then M admits an embedding to $\mathbb{R}^{2m+1}$.

a) Show that it is enough to prove b).

b) If $M \subseteq \mathbb{R}^N$ is a smooth submanifold and $N > 2m+1$, then there exists a hyperplane $H \subseteq \mathbb{R}^N$ such that the orthogonal projection to H restricts to an embedding from M to H.

   Hint: Refer to [Lafontaine, Cor. 3.8].

Exercise F6 (Level sets)

Let $F\colon \mathbb{R}^2 \to \mathbb{R}$ be defined by $F(x,y) = x^3 + xy + y^3$. Which level sets of F are embedded submanifolds of $\mathbb{R}^2$? Prove your answers.