Manifolds
Exercise Sheet 3.

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Groupwork

Exercise G1 (Hopf fibration)

Consider the 1-sphere and 3-sphere as subsets of the complex space, that is

$$S^1 = \{z \in \mathbb{C} : |z|^2 = 1\} \quad \text{and} \quad S^3 = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 = 1\}.$$

a) Show that the action of S^1 on S^3 by multiplication, i.e., $z \cdot (z_1, z_2) = (zz_1, zz_2)$, is a free action by homeomorphisms.

b) Show that there is a natural bijection $S^3/S^1 \approx \mathbb{C}P^1$.

c) Show that $\mathbb{C}P^1$ is diffeomorphic to S^2.

d) *Optional.* Show that the projection $\pi : S^3 \to S^3/S^1 \approx S^2$ is a smooth map.

In fact, it is a principal fiber bundle, called the Hopf fibration.

Exercise G2 (Tangent bundle to S^1)

Show that $\mathrm{T}S^1$ is a cylinder.

More precisely, show that $\mathrm{T}S^1 \approx S^1 \times \mathbb{R}$ *as smooth manifolds, in fact as smooth vector bundles.*

Exercise G3 (Coordinate vectors)

Let M be a smooth manifold and $(x^1, \ldots, x^m)$ be local coordinates. Denote (U, φ) the associated chart.

a) Recall the definition of the coordinate vectors $\dfrac{\partial}{\partial x^i}$.

Explain why any $v \in \mathrm{T}_p M$ is uniquely written $v = \displaystyle\sum_{i=1}^m v^i \frac{\partial}{\partial x^i}$ where $v^i = (\mathrm{d}x^i)_{|p}(v)$.

b) Show that alternatively, $\dfrac{\partial}{\partial x^i} = (\varphi_*)^{-1}(e_i)$ where $(e_i)_{1\leqslant i\leqslant m}$ is the canonical basis of $\mathbb{R}^m$.

What is the Jacobian matrix of $\varphi\colon U \to \mathbb{R}^m$, taking the local coordinates (x^i) on U and the usual coordinates on $\mathbb{R}^m$?

Exercise G4 (Jacobian matrix and change of coordinates)

How does the Jacobian matrix of a smooth map behave under changes of coordinates?

Homework

Hand in your work by Tuesday, 02.06.2020.

Exercise H1 (The Lie group $\mathrm{U}(1)$) **14 points**

a) Show that $\mathrm{U}(1)$ is:

- a matrix group, by identifying it to a subgroup of $\mathrm{GL}(2,\mathbb{R})$,
- a manifold, by identifying it to $S^1 \subseteq \mathbb{R}^2$.

Derive that $\mathrm{U}(1)$ is a Lie group.

b) Show that $\mathrm{SO}(2,\mathbb{R})$ is a matrix Lie group isomorphic to $\mathrm{U}(1)$.

You may admit that $\mathrm{SO}(2,\mathbb{R})$ *is a submanifold of* $\mathrm{GL}(2,\mathbb{R})$.

c) Show that $\mathbb{R}/\mathbb{Z}$ is a Lie group isomorphic to $\mathrm{U}(1)$.

Exercise H2 (Differentials) **8 points**

a) Let $f\colon M \to N$ and $g\colon N \to P$ be smooth maps between smooth manifolds.

- Show that $g\circ f$ is smooth, and express $\mathrm{d}(g\circ f)$ in terms of $\mathrm{d}f$ and $\mathrm{d}g$.
- Assume $f\colon M \to N$ is a diffeomorphism. Express $\mathrm{d}\left(f^{-1}\right)$ in terms of $\mathrm{d}f$.

b) Let M and N be smooth manifolds and let $f\colon M \to N$.

- Show that if f is constant, then f is smooth and $\mathrm{d}f \equiv 0$.
- Is the converse true?

Exercise H3 (Jacobian matrix) **8 points**

Let $f\colon M \to N$ and be a smooth map between smooth manifolds.

Let $p \in M$. Let $\varphi = (x^1, \ldots, x^m)$ [resp. $\psi = (y^1, \ldots, y^n)$] be a smooth chart on M [resp. on N], whose domain U contains p [resp. whose domain V contains $f(p)$].

a) Show that the following definitions of the *Jacobian of f at p* are equivalent:

- $\mathrm{Jac}_p(f) = \left[\frac{\partial f^j}{\partial x^i}_{|p}\right]_{\substack{1\leqslant i\leqslant m\\ 1\leqslant j\leqslant n}}$ where $f^j = y^j \circ f$.
- $\mathrm{Jac}(f)$ is the matrix associated to the linear map $\mathrm{d}f_{|p}\colon \mathrm{T}_p M \to \mathrm{T}_{f(p)} N$ in the bases $\left(\frac{\partial}{\partial x^i}\right)_{1\leqslant i\leqslant m}$ of $\mathrm{T}_p M$ and $\left(\frac{\partial}{\partial y^j}\right)_{1\leqslant j\leqslant n}$ of $\mathrm{T}_{f(p)} N$.
- $\mathrm{Jac}(f)$ is the Jacobian matrix of the map $\psi \circ f \circ \varphi^{-1}\colon \varphi(U) \subseteq \mathbb{R}^m \to \mathbb{R}^n$.

b) Is the Jacobian matrix of f at p is *intrinsic*, in the sense that it is independent of the choice of the charts φ and ψ? Is the rank of the Jacobian intrinsic?

Further Exercises

Exercise F1 (Complex projective space)

Define the complex projective space $\mathbb{C}P^m$ analogously to the real projective space.

a) Show that $\mathbb{C}P^m$ is a smooth manifold of dimension $2m$.

b) Show that $\mathbb{C}P^m$ is compact. *Hint: Show that there is a continuous map $S^{2m+1} \to \mathbb{C}P^m$.*

c) How would you define a complex manifold? Show that $\mathbb{C}P^m$ is an example.

Exercise F2 (Connectedness of $\mathrm{GL}(n, \mathbb{R})$)

a) Show that $\det\colon \mathrm{M}_n(\mathbb{R}) \to \mathbb{R}$ is a continuous map. *Can you say better?*

b) Derive that $\mathrm{GL}(n, \mathbb{R})$ is not connected.

c) (**) Show that $\mathrm{GL}(n, \mathbb{R})$ has two connected components. *Hint: Polar decomposition. Start with the case $n = 2$.*