Manifolds
2. Exercise Sheet

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Groupwork

Exercise G1 (Quiz - max 10 minutes)

Which of the following are differential manifolds, manifolds with boundary or neither?

a) A point,

b) closed ball $\{x \in \mathbb{R}^n : \|x\|_2 \leq 1\}$,

c) closed cube $\{x \in \mathbb{R}^n : \|x\|_\infty \leq 1\}$,

d) the cylinder $[a,b] \times \mathbb{S}^1$,

e) the cylinder $\mathbb{R} \times \mathbb{S}^1$,

f) the closed upper half plane $\{(x,y) \in \mathbb{R}^2 : y \geq 0\}$,

g) the open upper half plane $\{(x,y) \in \mathbb{R}^2 : y > 0\}$,

h) zero set of a polynomial in $\mathbb{R}^2$ of degree 2.

Hints for solution:

a) 0-dim. manifold

b) differential manifolds with boundary,

c) manifold with boundary (not diff.),

d) differential manifold with boundary,

e) differential manifold (without boundary),

f) differential manifold with boundary,

g) differential manifold (without boundary),

h) depends on the chosen polynomial.

Exercise G2 (Differential structure on $\mathbb{R}$)

Let $r > 0$ and define $\varphi_r \colon \mathbb{R} \to \mathbb{R}$ by

$$\varphi_r(x) = \begin{cases} x & x \leq 0, \\ rx & x > 0. \end{cases}$$

a) Show that the atlases $\{(\mathbb{R}, \varphi_r)\}_{r>0}$ define an uncountable family of pairwise distinct differentiable structures $\{\varphi_r : r > 0\}$ on $\mathbb{R}$.

b) Prove that the respective differentiable manifolds are pairwise diffeomorphic.

Hints for solution:

a) Since φ_r is a homeomorphism, $\{(\mathbb{R}, \varphi_r)\}$ is an atlas. The structures are different since for $r \neq s$ the transition map $\varphi_r \circ \varphi_s^{-1}$ is not differentiable.

b) Define $\varphi\colon \{\mathbb{R}, \varphi_r\} \to \{\mathbb{R}, \varphi_s\}$ as

$$\varphi(x) = \begin{cases} x & x \leq 0 \\ \frac{r}{s}x & x > 0. \end{cases}$$

Then, $\varphi_r \circ \varphi \circ \varphi_s^{-1}$ is the identity, therefore differentiable.

Exercise G3 (Change of coordinates: linear example)

Let V be a real vector space of dimension $n \in \mathbb{N}$.

a) Show that the choice of a basis of V yields a chart $\varphi\colon V \to \mathbb{R}^n$, which defines a smooth structure on V. What is the system of coordinates defined by φ?

b) If a different basis of V is chosen, what is the transition function produced by the change of charts? Describe the change of coordinates.

c) Conclude that the smooth structure on V defined by the choice of any basis is always the same.

Homework

Hand in your solutions until Tuesday, Mai 19^{th}.

Exercise H1 (Projective space) **12 points**

In this exercise we want to prove that the projective space $\mathbb{R}P^n$ is a differentiable manifold. First, recall the definition of projective space $\mathbb{R}P^n$ given in the lecture.

a) Show that $\mathbb{R}P^n$ is a topological manifold.

To introduce a differential structure we use *affine charts*. These are the homeomorphisms

$$\varphi_i : U_i := \{[u] : u_i \neq 0\} \subset \mathbb{R}P^n \to \mathbb{R}^n, \quad \varphi_i([u]) := \frac{1}{u_i}(u_1, ..., \widehat{u}_i, ..., u_{n+1}) \tag{1}$$

for $i = 1, ..., n+1$. The hat $\widehat{\cdot}$ indicates an entry to be omitted.

b) Show that the maps φ_i are indeed homeomorphisms.

c) Show that $\mathcal{A} := \{(\varphi_i, U_i) : i = 1, ..., n+1\}$ is a differential atlas for $\mathbb{R}P^n$.

d) Show that $\mathbb{R}P^n$ and $\mathbb{S}^n/\{\pm \operatorname{id}\}$ are homeomorphic.

Hints for solution:

a) Confirm Hausdorff, second countable and locally Euclidean by direct computation or topological arguments.

b) The inverse is given by

$$\varphi_i^{-1}\colon \mathbb{R}P^n \to U_i, \quad \varphi_i^{-1}(u_1,\dots,u_n) := [u_1,\dots,u_{i-1},1,u_i,\dots,u_n].$$

Indeed,

$$(\varphi_i\circ\varphi_i^{-1})(u) = \varphi_i\big([u_1,\dots,u_{i-1},1,u_i,\dots,u_n]\big) = \frac{1}{1}(u_1,\dots,u_{i-1},u_i,\dots,u_n) = u \quad \forall u\in\mathbb{R}P^n,$$
$$(\varphi_i^{-1}\circ\varphi_i)([u]) = \varphi_i^{-1}\Big(\frac{1}{u_i}(u_1,\dots,\widehat{u_i},\dots,u_{n+1})\Big) = \Big[\frac{u_1}{u_i},\dots,1,\dots,\frac{u_{n+1}}{u_i}\Big] = [u] \qquad \forall u\in U_i.$$

Both φ_i and φ_i^{-1} are continuous. Thus, φ_i is a homeomorphism.

c) We claim that the collection of our charts, $\mathcal{A} := \{(x_i,U_i) : i = 1,\dots,n+1\}$, forms an atlas. Clearly, the U_i cover $\mathbb{R}P^n$. Moreover, $\mathcal{A}$ is differentiable, since for $j<i$ and all $u\in\varphi_i\big(U_i\cap U_j\big) = \{u\in\mathbb{K}^n : u_i\neq 0 \text{ and } u_j\neq 0\}$

$$\big(\varphi_j\circ\varphi_i^{-1}\big)(u) = \varphi_j\big([u_1,\dots,u_{i-1},1,u_i,\dots,u_n]\big) = \frac{1}{u_j}(u_1,\dots,\widehat{u_j},\dots u_{i-1},1,u_i,\dots,u_n).$$

Similarly for $j>i$. This proves differentiability of the transition maps.

d) Consider the continuous map

$$\psi : \mathbb{R}P^n \to \mathbb{S}^n/\{\pm\,\mathrm{id}\}, \quad [u]\mapsto \operatorname{sgn}(u_{n+1})\,\frac{u}{\|u\|}.$$

Its inverse is given by

$$\psi^{-1} : \mathbb{S}^n/\{\pm\,\mathrm{id}\} \to \mathbb{R}P^n, \quad u = (u_1,\dots,u_{n+1}) \mapsto [u].$$

Indeed,

$$(\psi^{-1}\circ\psi)([u]) = \frac{\operatorname{sgn}(u_{n+1})}{\|u\|}\,[u_1,\dots,u_{n+1}] = [u_1,\dots,u_{n+1}] = [u].$$

and ψ^{-1} is continuous. Thus ψ is a homeomorphism from $\mathbb{R}P^n$ to $\mathbb{S}^n/\{\pm\,\mathrm{id}\}$.

Exercise H2 (Diffeomorphic smooth structures) **13 points**

Let M be a topological manifold of dimension $n\geqslant 1$.

a) Let $f\colon M\to M$ be a homeomorphism. If $\mathcal{A} = (U_i,\varphi_i)_{i\in I}$ is a smooth atlas on M, show that $f^*\mathcal{A} \coloneqq (f^{-1}(U_i),\varphi_i\circ f)_{i\in I}$ is a new smooth atlas on M (called *pullback* of $\mathcal{A}$ by f).

b) Show that $\mathcal{A}$ and $f^*\mathcal{A}$ are compatible smooth atlases if and only if f is a diffeomorphism of $(M,\mathcal{A})$.

c) (Bonus question) Show or admit that M admits a homeomorphism $f\colon M \to M$ that is not a diffeomorphism $(M, \mathcal{A}) \to (M, \mathcal{A})$.
Hint: construct such a homeomorphism that is the identity map outside a small open set.

d) Derive from the previous question any smooth manifold of dimension $\geqslant 1$ admits several incompatible smooth structures.

e) Show that any homeomorphism $f\colon M \to M$ is a diffeomorphism $(M, f^*\mathcal{A}) \to (M, \mathcal{A})$. Conclude that the incompatible smooth structures of the previous question are diffeomorphic.

Remark: Some manifolds, such as $\mathbb{R}^4$ and $\mathbb{S}^7$, admit several smooth structures that are not diffeomorphic.

Hints for solution:

a) Since the U_i cover M and f is a homeomorphism, the $f^{-1}(U_i)$ cover M as well. Since φ_i and f are homeomorphisms, $\varphi_i \circ f$ is a homeomorphism with inverse map $f^{-1} \circ \varphi_i^{-1}$ for all $i \in I$. For the transition maps it holds

$$(\varphi_j \circ f) \circ (f^{-1} \circ \varphi_i^{-1}) = \varphi_j \circ \varphi_i^{-1}.$$

Thus, they are smooth and $f^*\mathcal{A}$ is a smooth atlas.

b) Consider the transition maps $\varphi_j \circ (f^{-1} \circ \varphi_i^{-1})$ and $(\varphi_i \circ f) \circ \varphi_j^{-1}$. Since φ_i and φ_j are differentiable compatible, these transition maps are differentiable if and only if f and f^{-1} is differentiable, that is f is a diffeomorphism.

c) Bonus question

d) By c) M admits a homeomorphism f that is not a diffeomorphism. Then $\mathcal{A}$ and $f^*\mathcal{A}$ are not compatible by b). Hence, M has incompatible smooth structures.

e) Let $f : (M, f^*\mathcal{A}) \to (\mathcal{M}, \mathcal{A})$ be a homeomorphism and $p \in (M, f^*\mathcal{A})$. Let be $(f^{-1}(U_i), \varphi_i \circ f)$ a chart at p and (U_j, φ_j) a chart at $f(p)$. Then the locally defined function

$$(\varphi_j) \circ f \circ (f^{-1} \circ \varphi_i^{-1}) = \varphi_j \circ \varphi_i^{-1}$$

is differentiable. Hence, f is differentiable. Show analogously that f^{-1} is differentiable.

Further Exercises

Exercise F1 (Product manifolds)

a) Prove that if M_1 and M_2 are differential manifolds of dimension n_1 and n_2, resp., the product $M_1 \times M_2$ is a differential manifold of dimension $n_1 + n_2$.

b) Prove that if M_1 is a differential manifold and M_2 is a manifold with boundary, the product is a manifold with boundary.

c) What can we say about the product of two manifolds with boundary?

Exercise F2 (Example of coordinates on $\mathbb{S}^2$)

Let $\mathbb{S}^2$ be the unit sphere in $\mathbb{R}^3$.

a) Show that the spherical coordinates ϑ (the polar angle or colatitude) and φ (the azimuthal angle or longitude) define local coordinates on $\mathbb{S}^2$. Specifically, find one (or more) maximal domain $U \subseteq \mathbb{S}^2$ such that the map $(\vartheta, \varphi)\colon U \to \mathbb{R}^2$ is a well-defined chart.

b) Let (x, y, z) denote the usual Cartesian coordinates on $\mathbb{R}^3$. Show that (x, y) defines a system of local coordinates on $\mathbb{S}^2$. Specifically, find one (or more) maximal domain $U \subseteq \mathbb{S}^2$ such that the map $(x, y)\colon U \to \mathbb{R}^2$ is a well-defined chart. Same question for (x, z) and (y, z).

c) Describe the change of coordinates associated to all the charts discussed in the previous questions. Have you defined a smooth structure on $\mathbb{S}^2$?