Manifolds
1. Exercise Sheet

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Summer term 2020
24.04.2020

Groupwork

Exercise G1 (Euclidean topology on $\mathbb{R}^n$)
Consider $\mathbb{R}^n$ with the Euclidean topology $\mathcal{O}_{\mathrm{E}}$. That is the topology induced by the Euclidean metric:

$$U \subseteq \mathbb{R}^n \text{ open} \quad :\Longleftrightarrow \quad \forall\, x \in U \ \ \exists\, \varepsilon > 0 : B_\varepsilon(x) \subseteq U.$$

a) Show that $\mathcal{O}_{\mathrm{E}}$ is a topology on $\mathbb{R}^n$.

b) Show that $(\mathbb{R}, \mathcal{O}_{\mathrm{E}})$ is Hausdorff.

c) Show that $(\mathbb{R}, \mathcal{O}_{\mathrm{E}})$ is second-countable.

Hints for solution:

a) The condition obviously holds for the empty set and the whole space. Hence, $\emptyset, \mathbb{R}^2 \in \mathcal{O}_{\mathrm{E}}$.

Let $(U_i)_{i \in I}$ be a family of open set and V their union. Then for $x \in V$ there exists an $i_0 \in I$ such that $x \in U_{i_0}$. Then there exists an $\varepsilon_{i_0} > 0$ such that $B_{\varepsilon_{i_0}}(x) \subseteq U_{i_0} \subseteq V$. Thus, V is open.

Now suppose that I is finite, and let W be the intersection of the U_i. If $x \in U$, then $x \in U_i$ and there are $\varepsilon_i > 0$ such that $B_{\varepsilon_i}(x) \subseteq U_i$ for all $i \in I$. Then for $\varepsilon \coloneqq \min \varepsilon_i$ we have $B_\varepsilon(x) \subseteq U$ and U is open.

b) Let be $x, y \in \mathbb{R}^n$. Then for $\varepsilon \coloneqq \frac{1}{3} \|x - y\|$ the open sets $B_\varepsilon(x)$ and $B_\varepsilon(y)$ are disjoint.

c) The balls with rational midpoint and rational radii are a countable base.

Exercise G2 (Topological Manifolds)
Let M_1 and M_2 be topological manifolds. Discuss without proof whether the following sets are topological manifolds. Consider the sets in d) to f) as subsets of $\mathbb{R}^2$ with the Euclidean topology $\mathcal{O}_{\mathrm{E}}$.

a) $M_1 \cap M_2$,	d) $\{x^2 + y^2 = 1\}$,	g) $(\mathbb{R}^n, \mathcal{O}_{\mathrm{E}})$,
b) $M_1 \cup M_2$,	e) $\{x^2 - y^2 = 1\}$,	h) $(\mathbb{R}^n, \mathcal{O}_1)$ for $\mathcal{O}_1 := \{\text{all subsets of } \mathbb{R}^n\}$,
c) $M_1 \times M_2$,	f) $\{x^2 - y^2 = 0\}$,	i) $(\mathbb{R}^n, \mathcal{O}_2)$ for $\mathcal{O}_2 := \{\emptyset, \mathbb{R}^n\}$.

Hints for solution:

a) No, dimension can vary.

b) No.

c) Yes.

d) Yes, $\mathbb{S}^1$ is a topological manifold.

e) Yes, each sheet of this two-sheeted hyperbola can be regarded a graph over $(0, \infty)$.

f) No, in a neighbourhood of 0 the set does not admit a graph representation.

g) Yes, see G1.

h) No, the open sets are not second-countable.

i) No, not Hausdorff.

Exercise G3 (Stereographic projection)
Let $\mathbb{S}^n := \{x \in \mathbb{R}^{n+1} : \langle x, x \rangle = 1\}$ be the n-sphere with radius 1 where $\langle \cdot, \cdot \rangle$ is the standard scalar product on $\mathbb{R}^{n+1}$.

a) Is it possible to construct an atlas on $\mathbb{S}^n$ with only one chart?

Take the *North pole* $N := (0, 1) \in \mathbb{S}^1$ and define a map $x : \mathbb{S}^1 \to \mathbb{R}^1$ such that $x(p)$ is the intersection point of the x-axis with the straight line through N and p. This map is called *stereographic projection*.

b) Construct an atlas with the least number of charts on $\mathbb{S}^1$ using the stereographic projection.

c) Generalize this construction to $\mathbb{S}^n$.

Hints for solution:

a) No, since the chart would be a homeomorphism of the compact space $\mathbb{S}^n$ onto the non-compact space $\mathbb{R}^n$.

b) see part c).

c) $\mathcal{A} = \{(U_+, x_+), (U_-, x_-)\}$ where $U_\pm := \mathbb{S}^n \backslash \{N_\pm\}$ with $N_\pm := (0, ..., 0, \pm 1)$ and $x_+ \colon U_\pm \to \mathbb{R}^n$ given by

$$(x_1, \ldots, x_{n+1}) \mapsto \left(\frac{x_1}{1 \mp x_{n+1}}, \ldots, \frac{x_n}{1 \mp x_{n+1}} \right).$$

The charts are bijective: Indeed,

$$x_\pm^{-1} \colon \mathbb{R}^n \to U_\pm, \quad x_\pm^{-1}(u) := \frac{1}{|u|^2 + 1} \Big(2u, \, \pm(|u|^2 - 1) \Big)$$

are inverses to $x_\pm$. The charts and their inverses are continuous with respect to the relative topology: The coordinate functions are continuous functions, and so is their insertion into continuous functions. Finally, the two transition maps, $x_\pm \circ x_\mp^{-1}$ which map $x_\mp(U_+ \cap U_-) = \mathbb{R}^n \setminus \{0\}$ into itself,

$$\begin{aligned}\big(x_\pm \circ x_\mp^{-1}\big)(u) &= x_\pm\left(\frac{2u}{|u|^2+1},\ \mp(1-\frac{2}{|u|^2+1})\right)\\ &= \frac{1}{1+(1-\frac{2}{|u|^2+1})}\cdot\frac{2u}{|u|^2+1} = \frac{1}{2-\frac{2}{|u|^2+1}}\cdot\frac{2u}{|u|^2+1} = \frac{u}{|u|^2},\end{aligned}$$

are differentiable.

Homework

Hand in your solutions until Tuesday, Mai 5^{th}.

Exercise H1 (Alexandroff compactification) **10 points**

Let $n \in \mathbb{N}$ and $\widehat{\mathbb{R}}^n := \mathbb{R}^n \cup \{\infty\}$, where $\infty \notin \mathbb{R}^n$. Define open sets by

$$\mathcal{O} := \mathcal{O}_E \cup \mathcal{O}_\infty := \{U \subset \mathbb{R}^n : U \text{ is open in } \mathbb{R}^n\} \cup \{\widehat{\mathbb{R}}^n \backslash K : K \text{ is compact in } \mathbb{R}^n\}.$$

a) Show that $(\widehat{\mathbb{R}}^n, \mathcal{O})$ is a topological space.

b) Prove that $(\widehat{\mathbb{R}}^n, \mathcal{O})$ is a topological manifold.

c) Show that $\widehat{\mathbb{R}}^n$ is homeomorphic to $\mathbb{S}^n$.

Hints for solution:

a) We have $\emptyset \in \mathcal{O}_E$ and since $\emptyset$ is compact $\widehat{\mathbb{R}}^n = \widehat{\mathbb{R}}^n \backslash \emptyset \in \mathcal{O}_\infty$.

Let $(U_i)_{i\in I}$ be a family of open sets, and let V be their union. If none of the U_i contains ∞, then neither does V. Hence V is open in $\widehat{\mathbb{R}}^n$ since it is open in $\mathbb{R}^n$. If at least one of them, say U_{i_0}, contains ∞, then let $K := \mathbb{R}^n \backslash U_{i_0} = \widehat{\mathbb{R}}^n \backslash U_{i_0}$. Now,

$$\mathbb{R}^n \backslash V = \bigcap_{i\in I} (\mathbb{R}^n \backslash U_i) \subseteq K,$$

so $\mathbb{R}^n \backslash V$ is closed and contained in K. Since K is also closed, $\mathbb{R}^n \backslash V$ is closed in K as well. Hence $\mathbb{R}^n \backslash V$ is compact as closed subsets of compact sets are compact.

Now suppose that I is finite, and let W be the intersection of the U_i. If all of the U_i contain ∞, then so does their intersection. Since $\mathbb{R}^n \backslash W$ is a finite union of compact spaces, it is compact, so W is open. If one of the U_i does not contain ∞, then $\infty \notin W$, and we can write W as a finite intersection of open sets of $\mathbb{R}^n$.

b) Let $p, q \in \widehat{\mathbb{R}}^n, p \neq q$. If $p \neq \infty \neq q$, we can use two disjoint open sets in $\mathbb{R}^n$ to separate p and q. If one of them is equal to ∞, say q, take a closed ball with radius r_1 with center p and an open ball with radius $r_2 < r_1$ with center p. Then, the complement of the closed ball and the open ball are two disjoint open subsets separating p and q. Hence, $\widehat{\mathbb{R}}^n$ is Hausdorff. Second countability is obvious. To show that $\widehat{\mathbb{R}}^n$ is locally homeomorphic to Euclidean space, construct suitable homeomorphisms (cf. part c)).

c) Since $\mathbb{S}^n$ is compact and $\mathbb{S}^n \backslash \{p\}$ is dense in $\mathbb{S}^n$, it suffices to show that $\mathbb{R}^n$ is homeomorphic to $\mathbb{S}^n \backslash \{p\}$. The stereographic projection is the desired homeomorphism. Hence, $\widehat{\mathbb{R}}^n$ is homeomorphic to $\mathbb{S}^n$.

Exercise H2 (Non-Hausdorff space) **7 points**

Let $L := \{0, 1\} \times \mathbb{R}$. Define on L an equivalence relation $\sim$ by

$$(0, y) \sim (1, y) \iff y \neq 0.$$

a) What are the equivalence classes on the quotient set $L/\sim$?

b) Show that $L/\sim$ is locally Euclidean (locally homeomorphic to $\mathbb{R}$).

c) Show that $L/\sim$ is not Hausdorff.

Hints for solution: The quotient set $L/\sim$ is a line with double origin. To show that it is locally homeomorphic to $\mathbb{R}$ take the projection onto $\mathbb{R}$. The quotient $L/\sim$ is not Hausdorff in the double origin, since for every open neighborhoods of $(0, 0)$ and $(1, 0)$ there exists an $\varepsilon > 0$ such that $(0, \varepsilon)$ and $(1, \varepsilon)$ are contained in the neighborhoods, resp., but $(0, \varepsilon) \sim (1, \varepsilon)$. Thus, the neighborhoods are not disjoint.

Exercise H3 (Lie groups) **13 points**

A *Lie group* G is a topological manifold which is a group such that multiplication $(g, h) \to gh$ and inversion $g \to g^{-1}$ are continuous, where $G \times G$ is equipped with the product topology.

Let $\mathsf{M}(n, \mathbb{R})$ be the set of all real $n \times n$ matrices and $\mathsf{GL}(n, \mathbb{R})$ the subset of all invertible $n \times n$ matrices. We may identify $\mathsf{M}(n, \mathbb{R})$ with $\mathbb{R}^{n^2}$ and $\mathsf{GL}(n, \mathbb{R})$ as a subset of $\mathbb{R}^{n^2}$. We assume $\mathsf{M}(n, \mathbb{R})$ to be equipped with the Euclidean topology of $\mathbb{R}^{n^2}$ and $\mathsf{GL}(n, \mathbb{R})$ to be equipped with the induced subset topology.

a) Show that $(\mathsf{M}(n, \mathbb{R}), +)$ is a Lie group.

b) Show that the determinant $\det : \mathsf{M}(n, \mathbb{R}) \to \mathbb{R}$ is a continuous map.

c) Show that $(\mathsf{GL}(n, \mathbb{R}), \cdot)$ is a Lie group.

A *homomorphism of Lie groups* is a continuous group homomorphism.

d) Is the exponential map $\exp : \mathsf{M}(n, \mathbb{R}) \to \mathsf{GL}(n, \mathbb{R})$ a homomorphism of Lie groups?

Hints for solution:

a) $\mathsf{M}(n,\mathbb{R})$ is a topological manifold by G1. Matrix addition is associative, for $A \in \mathsf{M}(n,\mathbb{R})$ we have $-A \in \mathsf{M}(n,\mathbb{R})$ and the zero matrix is also contained in $\mathsf{M}(n,\mathbb{R})$. Hence, $\mathsf{M}(n,\mathbb{R})$ is a group. Since addition and subtraction of matrices are continuous, $(\mathsf{M}(n,\mathbb{R}),+)$ is a Lie group.

b) Since $\det(A)$ is a polynomial in the entries of A it is continuous.

c) Since matrix multiplication is a associative, the matrices in $\mathsf{GL}(n,\mathbb{R})$ are invertible by choice and the identity is in $\mathsf{GL}(n,\mathbb{R})$, we have that $(\mathsf{GL}(n,\mathbb{R}),\cdot)$ is a group. Since $\det$ is continuous by (b) and $\det^{-1}(\mathbb{R}\backslash\{0\}) = \mathsf{GL}(n,\mathbb{R})$, we have that $\mathsf{GL}(n,\mathbb{R})$ is an open subset of $\mathsf{M}(n,\mathbb{R})$ and hence homeomorphic to $\mathbb{R}^{n^2}$. Thus, $\mathsf{GL}(n,\mathbb{R})$ is a topological manifold. Matrix multiplication is continuous since the entries of AB are polynomials in the entries of A and B. The inverse of a matrix A is given by $(A^{-1}_{ij}) = (-1)^{i+j}\det(A^{ji})/\det(A)$, where A^{ji} is the matrix A after deleting the j-th row and i-th column. Since the determinant is continuous, we get that the inversion map is continuous. Hence, $(\mathsf{GL}(n,\mathbb{R}),\cdot)$ is a Lie group.

d) Since $\exp(A+B) = \exp(A)\exp(B)$ if and only if A and B commute, $\exp$ is not a group homomorphism for $n > 1$.

Further Exercises

These additional exercises are not compulsory.

Exercise F1 (Connected sum)
Let M, N be two n-dim. manifolds. The connected sum $M \sharp N$ is the manifold formed by deleting a ball inside each manifold and gluing together the resulting boundary spheres, see Figure 1.

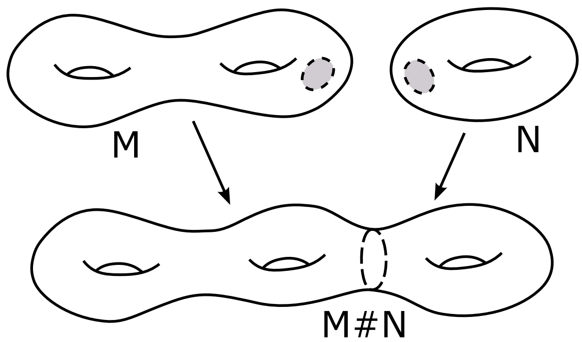

Figure 1: Connected sum $M \sharp N$. (https://en.wikipedia.org/wiki/Connected_sum)

We want to show some properties of the connected sum. It suffices to give reasonable arguments. A precise proof is not required.

a) Show that the connected sum is invariant under homeomorphisms, i.e., if $M_1 \approx M_2$ and $N_1 \approx N_2$ then $M_1 \sharp N_1 \approx M_2 \sharp N_2$.

b) Show that the connected sum is associative (up to homeomorphisms).

c) Show that the sphere is the neutral element, i.e., $M^n \sharp \mathbb{S}^n \approx M^n$.

Let $S_g := T^2 \sharp T^2 \sharp \cdots \sharp T^2$ be the connected sum of g tori.

d) Show that S_g is a connected closed manifold and well-defined up to homeomorphisms.

e) Show that if $S_g = S_{g'}$ then $g = g'$.

Remark: The connected sum is important for the classification of surfaces. In the 1-dimensional case every connected manifold is either homeomorphic to $\mathbb{R}^1$ or $\mathbb{S}^1$. In the 2-dimensional case every connected closed manifold is either homeomorphic to the connected sum of g tori or the connected sum of k projective planes. In the first case the surface is orientable and has Euler characteristic $2 - 2g$. In the second case the surface is non-orientable and has Euler characteristic $2 - k$. To show this statement is not part of this class.

Exercise F2 (Manifolds not satisfying the "mild topological restrictions")
Find examples of topological spaces which are locally homeomorphic to Euclidean space but not Hausdorff and/or second-countable.

Hints for solution:

- Not Hausdorff: line with two origins, see H2
- Not second-countable: disjoint union of uncountably many copies of $\mathbb{R}$

Exercise F3 (Connectedness and Path-Connectedness)
Prove that a connected manifold is path-connected.