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Hyperbolic Geometry

Winter 2019-2020

Exercise Sheet 7 (Chapter 9, 10, 11)

Chapter 9

Exercise 1. (*) Quasi-isometric spaces

Let (X, d_X) and (Y, d_Y) be metric spaces. A map $f\colon X \to Y$ is called a *quasi-isometry* if:

(i) f is coarsely Lipschitz: there exists $A \geqslant 1, B \geqslant 0$ such that for all $x_1, x_2 \in X$:

$$\frac{1}{A} d_X(x_1, x_2) - B \leqslant d_Y(f(x_1), f(x_2)) \leqslant A d_X(x_1, x_2) + B\ .$$

(ii) f is coarsely surjective: there exists $C \geqslant 0$ such that for all $y \in Y$, there exists $x \in X$ such that $d(f(x), y) \leqslant C$.

When there exists a quasi-isometry $f\colon X \to Y$, one says that the metric spaces X and Y are *quasi-isometric*.

(1) Show that any metric space of finite diameter is quasi-isometric to a point.

(2) Show that $\mathbb{R}^2$ and $\mathbb{H}^2$ are not quasi-isometric.

(3) Show that any quasi-isometry $f\colon \mathbb{H}^m \to \mathbb{H}^n$ extends to a homeomorphism $\partial_\infty \mathbb{H}^m \to \partial_\infty \mathbb{H}^n$. Conclude that $\mathbb{H}^m$ is quasi-isometric to $\mathbb{H}^n$ if and only if $m = n$.

Exercise 2. Ideal boundary of the hyperboloid model and the Cayley-Klein model

We identified both the ideal boundary of the hyperboloid model $\mathcal{H}^+ \subseteq \mathbb{R}^{n,1}$ and the ideal boundary of the Cayley-Klein model $\Omega^- \subseteq \mathbf{P}(\mathbb{R}^{n,1})$ as the projectivized light cone of $\mathbb{R}^{n,1}$. Can you explain this “coincidence”?

Exercise 3. Busemann function in the Poincaré disk

Let $X = (B^2, g_{B^2})$ be the Poincaré disk. We use the complex coordinate z on the unit disk $\mathbb{D} \approx B^2$.

(1) For any $\xi \in \partial_\infty X = \{z \in \mathbb{C} \mid |z| = 1\}$, check that the geodesic ray $r_\xi\colon [0, +\infty) \to X$ such that $r(0) = 0$ and $r(+\infty) = \xi$ has the expression: $r(t) = \tanh(t/2)\, \xi$.

(2) Show that the Busemann function B_r is given by

$$B_r(z) = -\ln\left(\frac{1 - |z|^2}{|z - \xi|^2}\right)\ .$$

(3) Recover the fact that horocycles centered at ξ are Euclidean circles tangent to $\partial_\infty X$ at ξ.

Exercise 4. Horospheres as limit of spheres

Let $x_0 \in \mathbb{H}^n$ and let $P \subseteq \mathrm{T}_{x_0} \mathbb{H}^n$ be a hyperplane.

(1) Show that for all $r > 0$, there exists exactly two hyperspheres $S_1(r)$ and $S_2(r)$ in $\mathbb{H}^n$ that go through x_0 and are tangent to P.

(2) Show that there exists exactly two horospheres S_1 and S_2 in $\mathbb{H}^n$ that go through x_0 and are tangent to P.

(3) Show that $\{\lim_{r\to+\infty} S_1(r), \lim_{r\to+\infty} S_2(r)\} = \{S_1, S_2\}$.

Exercise 5. Horospheres as hypersurfaces with asymptotic normal geodesics

(1) Let S be a horosphere centered at $\xi \in \partial_\infty \mathbb{H}^n$. Show that for any $x_0 \in S$, the geodesic going through x and with ideal endpoint ξ intersects S orthogonally. Show that it is also orthogonally transverse to any other horosphere centered at ξ.

(2) Show that a complete hypersurface $S \subseteq \mathbb{H}^n$ is a horosphere if and only if all geodesics that intersect S orthogonally share an ideal endpoint.

Exercise 6. Horospheres in the hyperboloid model

Show that in the hyperboloid model $\mathcal{H}^+ \subseteq \mathbb{R}^{n,1}$, horospheres are given by the intersection of $\mathcal{H}^+$ with hyperplanes of $\mathbb{R}^{n,1}$ whose normal lies in the light cone. Show that when $n = 2$, these are parabolas (also see Exercise 4.4).

Exercise 7. Horospheres in the Klein model

Show that in the Beltrami-Klein disk $B^2 \subseteq \mathbb{R}^2$, the horocycles centered at $\xi \in S^1$ are the Euclidean ellipses contained in B^2 that have a contact of order 4 with S^1 at ξ. Suggest and prove an analogous characterization in higher dimensions. Argue that this characterization also makes sense in the Cayley-Klein model.

Exercise 8. Isometries fixing an ideal point

Let $X = \mathbb{H}^n$ and $\xi \in \partial_\infty X$.

(1) Show that if $f \in \mathrm{Isom}(X)$ fixes ξ, then f maps any horosphere S centered at ξ to some other such horosphere S'. *Optional: in what case do we have $S' = S$?*

(2) Recall that any horosphere S is isometric to $\mathbb{R}^{n-1}$. Recall explicitly the isometric identification $S \approx \mathbb{R}^{n-1}$ when S is a horosphere centered at $\xi = \infty$ in the Poincaré half-space model. Show that f induces an affine similarity of $\mathbb{R}^{n-1}$.

(3) Recover the fact that the subgroup of the Möbius group of S^{n-1} fixing a point is isomorphic to the group of affine similarities of $\mathbb{R}^{n-1}$ (see Exercise 7.6).

Chapter 10

Exercise 1. Characterization of translation length (*borrowed from [**?**, Chap. II.6].*)

Let X be a metric space and let $f\colon X \longrightarrow X$.

(1) Show that for any $x \in X$, the sequence $\frac{1}{n} d(x, f^n(x))$ converges in $[0, +\infty)$. *Hint: First show that $d(x, f^n(x))$ is a sub-additive function of n. Then show that $\frac{g(n)}{n}$ converges for any sub-additive function $g\colon \mathbb{N} \longrightarrow \mathbb{R}$.*

(2) Show that $\lim_{n\to+\infty} \frac{1}{n} d(x, f^n(x))$ is independent of x.

(3) Show that if f is semi-simple (elliptic or hyperbolic), then $l_f = \lim_{n\to+\infty} \frac{1}{n} d(x, f^n(x))$.

Exercise 2. Parabolic fixed point

Let f be a parabolic isometry of $X = \mathbb{H}^n$. Denote $\xi \in \partial_\infty X$ its ideal endpoint.

(1) Show that for any $x \in X \cup \partial_\infty X$, $\lim_{n\to+\infty} f^n(x) = \xi$. Is ξ an attracting fixed point?

(2) Show that for any compact set $K \subseteq \partial_\infty X - \{\xi\}$ and for any neighborhood U of ξ in $\partial_\infty X$, $f^n(K) \subseteq U$ for n sufficiently large. Is ξ an attracting fixed point?

Exercise 3. Translation length of a parabolic

Let f be a parabolic isometry of $X = \mathbb{H}^n$. Show that f has zero translation length.

Exercise 4. Equidistant curves and translations

(1) Let $L \subseteq \mathbb{H}^n$ be a geodesic line. How would you define an equidistant curve from L? Show that for any $x_0 \in \mathbb{H}^n$, there exists a unique equidistant curve from L.

(2) Let L be the geodesic line with ideal endpoints 0 and ∞ in the Poincaré half-space H^n. Show that the equidistant curves from L are the Euclidean straight half-lines starting from 0.

(3) Prove Proposition 10.18: a map $f\colon \mathbb{H}^n \longrightarrow \mathbb{H}^n$ is a translation if and only if there exists an isometry $\varphi\colon \mathbb{H}^n \longrightarrow H^n$ such that $\varphi f \varphi^-1$ is given by $x \in H^n \mapsto e^l x$, where l is the translation length of f.

Exercise 5. Fixed points and trace

Recall Lemma 10.24: Let $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}(2, \mathbb{R})$ and denote $f\colon z \mapsto \frac{az+b}{cz+d}$ the associated fractional linear transformation of $\hat{\mathbb{C}}$.

- If $(\mathrm{tr}\, M)^2 > 4$, then f has two fixed points, both of which lie in $\hat{\mathbb{R}} \subseteq \hat{\mathbb{C}}$.
- If $(\mathrm{tr}\, M)^2 < 4$, then f has two fixed points, one in $\mathbb{H}$ and the other is its complex conjugate.
- If $(\mathrm{tr}\, M)^2 = 4$, then either f is the identity, or f has a unique fixed point, which lies in $\hat{\mathbb{R}} \subseteq \hat{\mathbb{C}}$.

(1) Prove the lemma by direct computation, solving the equation $\frac{az+b}{cz+d} = z$.

(2) Consider the projective transformation $\hat{f}\colon \mathbb{C}P^1 \to \mathbb{C}P^1$ associated to M. Explain why the fixed points of $\hat{f}$ are the eigenlines of M. Recover the lemma.

Exercise 6. Limits of loxodromics

(1) Recall the "standard form" of orientation-preserving elliptic, loxodromic, and parabolic isometries of $\mathbb{H}^3$ in the Poincaré half-space model.

(2) Using the previous question, show that any elliptic element of $\mathrm{Isom}^+(\mathbb{H}^3)$ can be obtained as a limit of loxodromic elements.

(3) Prove more generally that any elliptic isometry of $\mathbb{H}^n$ can be obtained as a limit of loxodromic isometries.

(4) Going back to $\mathbb{H}^3$, write a different proof using matrices. Prove in fact that loxodromic elements are dense in $\mathrm{Isom}^+(\mathbb{H}^3)$.

Exercise 7. A baby character variety

Let us work in the Poincaré half-space model $\mathbb{H} \subseteq \mathbb{C}$ of the hyperbolic plane $\mathbb{H}^2$. We denote $G = \mathrm{Isom}^+(\mathbb{H})$ the group of orientation-preserving isometries, which can be identified to $\mathrm{PSL}(2,\mathbb{R}) = \mathrm{SL}(2,\mathbb{R})/\{\pm I_2\}$ equipped with the quotient topology.

(1) Show that $f_0 = \mathrm{id} \in G$ is in the closure of the conjugacy class $C \subseteq \mathrm{Isom}^+(\mathbb{H})$ of some/any parabolic isometry.

(2) Let G act on itself by conjugation. Derive from the previous question that the quotient $\mathcal{R}$ is not Hausdorff.

(3) (*) We recall that an element of G is called *semisimple* (or *completely reducible*, or *polystable*, depending on context) if it is not parabolic. Let $\mathcal{X} \subseteq \mathcal{R}$ denote the subset of conjugacy classes of semisimple elements. Show that $\mathcal{X}$ is Hausdorff.

Exercise 8. Trace relations

We let $G = \mathrm{SL}(2,\mathbb{C})$ in this exercise.

(1) Show that for any $A, B \in G$, $\mathrm{tr}(AB) + \mathrm{tr}(AB^{-1}) = \mathrm{tr}\,A\,\mathrm{tr}\,B$.

(2) Show that the trace of any element of the subgroup of G generated by A and B can be expressed as a polynomial in $\mathrm{tr}\,A$, $\mathrm{tr}\,B$, and $\mathrm{tr}\,AB$ with integer coefficients.

(3) *Optional.* Show that any polynomial function of $(A, B) \in G \times G$ that is invariant by conjugation (that is, invariant by $(A, B) \mapsto (gAg^{-1}, gBg^{-1})$ for all $g \in G$) can be expressed as a polynomial function of $\mathrm{tr}\,A$, $\mathrm{tr}\,B$, and $\mathrm{tr}\,AB$.

Exercise 9. Classification in $\mathrm{O}^+(n,1)$

Recall that $\mathrm{Isom}(\mathbb{H}^n) \approx \mathrm{O}^+(n,1)$, e.g. via the hyperboloid model. Using linear algebra, find a characterization of elliptic, loxodromic, and parabolic elements of $\mathrm{O}^+(n,1)$.

Chapter 11

Exercise 1. Congruent triangles with ideal vertices

We have seen (Theorem 11.2) that two hyperbolic triangles are congruent if and only if they have the same side lengths. State and prove a generalization for triangles having one or more ideal vertices.

Exercise 2. Congruent triangles and angles

Show that for any three numbers $\alpha, \beta, \gamma \geqslant 0$ such that $\alpha + \beta + \gamma < \pi$, there exists a hyperbolic triangle whose interior angles are equal to α, β, γ. Show that moreover, any two such triangles are congruent. Is this true for Euclidean triangles?

Exercise 3. Inscribed and Circumscribed circles

(1) Show that not all hyperbolic triangles admit a circumscribed circle.

(2) In Chapter 10, we saw that any bounded set in $\mathbb{H}^n$ has a well-defined *minimum bounding ball*, which some authors call "circumball". Is that not a contradiction with the previous question?

(3) Show that any hyperbolic triangle admits a uniquely defined inscribed circle.

(4) Show that there exists a finite upper bound for the radii of the inscribed circle of all hyperbolic triangles.

Exercise 4. Unified law of cosines

For $R \in \mathbb{C} - 0$, define the generalized cosine and sine functions by:

$$\cos_R(x) = \cos\left(\frac{x}{R}\right)$$
$$\sin_R(x) = R\sin\left(\frac{x}{R}\right) .$$

Consider the "unified law of cosines for curvature $k = \frac{1}{R^2}$":

$$\cos_R c = \cos_R a \cos_R b + \frac{1}{R^2} \sin_R a \sin_R b \cos \hat{C} .$$

(1) Check that in the case $k = -1$, i.e. $R = \pm i$, one recovers the hyperbolic law of cosines.

(2) Prove the hyperbolic law of cosines in the hyperbolic space of constant curvature $k < 0$.

(3) Predict the spherical law of cosines. Prove it.

(4) Show that the Euclidean law of cosines may be obtained asymptotically from the unified law of cosines when $k \to 0$.

(5) *Optional.* Can you come up with a heuristic explanation for the existence of a unified law of cosines that works in any constant curvature?

Exercise 5. Area of hyperbolic polygons

How would you define a hyperbolic polygon? Find a formula for the area of any hyperbolic polygon, and prove it.

Exercise 6. Gromov hyperbolicity of hyperbolic space

Let $n \geqslant 2$. The goal of this exercise is to show that hyperbolic space $\mathbb{H}^n$ is Gromov hyperbolic (see Definition 9.8): there exists $\delta > 0$ such that any triangle in $\mathbb{H}^n$ is δ-thin.

(1) Argue that it is enough to do the case $n = 2$.

(2) Argue that it is enough to show that some ideal triangle is δ-thin.

(3) Consider the ideal triangle with vertices $A = 0$, $B = \infty$, and $C = 1$ in the Poincaré half-plane. What are the sides (AB), (BC), and (CA) of this triangle? Draw a picture.

(4) Let $p = (0, y) \in (AB)$. Show that the distance from p to (BC) is achieved at $p' = (1, \sqrt{1 + y^2})$. Derive that $d(p, (BC)) = \operatorname{arsinh}(1/y)$.

(5) Find an isometry that maps $A \mapsto B$, $B \mapsto C$, $C \mapsto A$. Show that $d(p, (CA)) = \operatorname{arsinh}(y)$.

(6) Conclude that $d(p, (BC) \cup (CA)) \leqslant \delta$ where $\delta = \operatorname{arsinh}(1)$ and conclude the exercise.