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Hyperbolic Geometry

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Exercise Sheet 5 (Chapter 7)

Chapter 7

Exercise 1. Characterization of conformal maps of $\mathbb{R}^n$.

Let V, W be Euclidean vector spaces and $\Omega \subseteq V$ be an open set. Consider an immersion $f\colon \Omega \to V$.

(1) Let γ_1 and γ_2 be two regular curves in Ω that intersect at $p \in \Omega$. Denote v_i the tangent vector to γ_i at p. Show that $f \circ \gamma_1$ and $f \circ \gamma_2$ are two regular curves in W that intersect at $f(p)$, and that the tangent vector to γ_i at $f(p)$ is $\mathrm{d}f(v_i)$.

(2) Prove Proposition 7.6: *Let $f\colon \Omega \subseteq V \to W = V$. Then f is conformal if and only if f is differentiable and $\mathrm{d}f_x$ is a linear similarity for all $x \in \Omega$.*

(3) Prove Proposition 7.7: *$f\colon \Omega \subseteq \mathbb{C} \to \mathbb{C}$ is conformal if and only if f is holomorphic or anti-holomorphic and f' does not vanish.* (This question requires basic knowledge of holomorphic functions.)

Exercise 2. Characterization of conformal maps between Riemannian manifolds

Let (M, g) and (N, h) be Riemannian manifolds.

(1) Let $f\colon V \to W$ be a linear map between vector spaces. For any bilinear form b on W, we define the bilinear form f^*b on V by $f^*b(u, v) := b(f(u), f(v))$. Show that if b is an inner product, f^*b is an inner product if and only if f is injective.

(2) Let $f\colon (V, \langle\cdot,\cdot\rangle_V) \to (W, \langle\cdot,\cdot\rangle_W)$ be a linear map between Euclidean vector spaces. Show that f is angle-preserving if and only if there exists $\lambda \in \mathbb{R}_{>0}$ such that $f^*\langle\cdot,\cdot\rangle_W = \lambda\langle\cdot,\cdot\rangle_V$.

(3) Let $f\colon (M, g) \to (N, h)$ be a differentiable map between Riemannian manifolds. How do you define the pullback f^*h? Show that f is conformal if and only if f^*h is conformal to g.

Exercise 3. Full vs restricted Möbius group

Denote $\mathrm{Möb}^+(S^n)$ the restricted Möbius group of S^n, consisting of orientation-preserving Möbius transformations.

(1) Show that $\mathrm{Möb}^+(S^n)$ is an index 2 normal subgroup of $\mathrm{Möb}(S^n)$.

(2) Show that $\mathrm{Möb}^+(S^n)$ is the identity component of $\mathrm{Möb}(S^n)$.

(3) Show the same results for $\mathrm{Möb}^+(B^n) < \mathrm{Möb}(B^n)$ and $\mathrm{Möb}^+(\widehat{\mathbb{R}^n}) < \mathrm{Möb}(\widehat{\mathbb{R}^n})$.

Exercise 4. Inversions

(1) Let $S = S(a, r)$ be the sphere of center a and radius r in $\mathbb{R}^n$. What is its Cartesian equation? Show that the inversion through S has the expression:

$$f(x) = a + \frac{r^2}{\|x - a\|^2}(x - a)\,.$$

(2) Let $P \subseteq \mathbb{R}^n$ be an affine hyperplane. Denote v a nonzero normal vector and $\lambda \in \mathbb{R}$ such that $x_0 = \lambda v$ belongs to P (why is λ well-defined?). Show that the Cartesian equation of P is $\langle x - x_0, v\rangle = 0$. Show that the inversion through P has the expression:

$$f(x) = x - 2\langle x - x_0, v\rangle \frac{v}{\|v\|^2}\,.$$

(3) Show that the results of (2) may be obtained by taking the limit of (1) with $a = x_0 + tv$ and $r = t\|v\|$ when $t \longrightarrow +\infty$.

(4) Recover the result that any finite product of inversions may be written

$$f(x) = b + \frac{\alpha A(x - a)}{|x - a|^\varepsilon}$$

where $a, b \in \mathbb{R}^n$, $\alpha \in \mathbb{R}$, $A \in \mathrm{O}(n)$, and $\varepsilon \in \{0, 2\}$.

Exercise 5. More inversions

(1) Show that any translation $\mathbb{R}^n \longrightarrow \mathbb{R}^n$ can be written as a product of two reflections. Could you expect such a result?

(2) Show that any linear similarity $\mathbb{R}^n \longrightarrow \mathbb{R}^n$ can be written as a product of two inversions. Could you expect such a result?

Exercise 6. Möbius transformations vs Euclidean similarities

Show that the subgroup of Möb($\widehat{\mathbb{R}^n}$) fixing ∞ is isomorphic to the group of affine similarities of $\mathbb{R}^n$.

Exercise 7. Stereographic projection

(1) Recover the expression of the standard stereographic projection $s\colon S^n \longrightarrow \widehat{\mathbb{R}^n}$.

(2) Recover that the stereographic projection is the restriction to S^n of an inversion of $\widehat{\mathbb{R}^{n+1}}$. Derive that s is a conformal equivalence.

(3) Recover that s is conformal by direct computation: compute the pullback Riemannian metric $s^* g$ on $S^n - \{N\}$, where g is the Euclidean metric on $\mathbb{R}^n$.

Exercise 8. Poincaré extension

(1) Find the Poincaré extension of an inversion of $\widehat{\mathbb{R}^n}$.

(2) Write a new proof of the existence of the Poincaré extension of a Möbius transformation. Can you extend your argument to also prove uniqueness?

Exercise 9. Möbius transformations of $\hat{\mathbb{C}}$

The goal of this exercise is to show Theorem 7.43: *A map $f\colon \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is an Möbius transformation if and only if it is fractional linear (orientation-preserving case) or its conjugate is fractional linear (orientation-reserving case).*

(1) Argue that it is enough to show that f is an orientation-preserving Möbius transformation if and only if it is fractional linear.

(2) (a) Show that the inversion through the sphere $S(a,r)$ can be written $f(z) = a + \frac{r^2}{\bar{z}-\bar{a}}$.
  (b) Show that the inversion through the line with normal vector v going through the point $z_0 = \lambda v$ can be written $f(z) = 2z_0 - \frac{v}{\bar{v}}\bar{z}$.
  (c) Show that the composition of any two inversions is fractional linear. Conclude that any Möbius transformation of $\hat{\mathbb{C}}$ is fractional linear.

(3) (a) Show that any fractional linear transformation may be written as a composition of maps of the form: $z \mapsto z + b$ where $b \in \mathbb{C}$, $z \mapsto az$ where $a \in \mathbb{C}^*$, and $z \mapsto \frac{1}{z}$.
  (b) Show that the three maps of the previous question may be written as a product of inversions.
  (c) Conclude that any fractional linear transformation is a Möbius transformation of $\hat{\mathbb{C}}$.

Exercise 10. The group $\mathrm{PSU}(1,1)$

(1) Recall the definition of $\mathrm{SU}(1,1)$ and show that

$$\mathrm{SU}(1,1) = \left\{ \begin{bmatrix} a & b \\ \bar{b} & \bar{a} \end{bmatrix} \in \mathcal{M}_{2\times 2}(\mathbb{C}) \mid |a|^2 - |b|^2 = 1 \right\}$$

(2) Show that $\mathrm{U}(1,1) = \{uA \mid |u| = 1, A \in \mathrm{SU}(1,1)\}$. Derive that $\mathrm{PU}(1,1) \approx \mathrm{PSU}(1,1)$.

(3) Show that the action of any element of $\mathrm{U}(1,1)$ by fractional linear transformation can be written

$$z \mapsto u\frac{z-a}{1-\bar{a}z}$$

where $|u| = 1$ and $a \in \mathbb{D}$.

(4) Recover from the previous question that the action of $\mathrm{U}(1,1)$ on $\hat{\mathbb{C}}$ preserves $\mathbb{D}$.

(5) Prove that conversely, a fractional linear transformation preserving $\mathbb{D}$ coincides with the action of an element of $\mathrm{U}(1,1)$.

(6) Recall why $\mathrm{Möb}^+(\mathbb{D}) \approx \mathrm{Aut}(\mathbb{D}) \approx \mathrm{PSU}(1,1)$.

Exercise 11. The group $\mathrm{PSL}(2,\mathbb{R})$

(1) Recover by direct proof that the Cayley transform $c(z) = i\,\frac{z-i}{z+i}$ defines a biholomorphism from $\mathbb{H}$ to $\mathbb{D}$.

(2) Recover by direct proof that the fractional linear action of $M \in \mathrm{SL}(2,\mathbb{C})$ on $\hat{C}$ preserves $\mathbb{H}$ if and only if M has real coefficients.

(3) Recover by direct proof that $\mathrm{SL}(2,\mathbb{R}) = C^{-1}\,(\mathrm{SU}(1,1))\,C$ where $C = \begin{bmatrix} i & 1 \\ 1 & i \end{bmatrix}$. Recall the connection between this result and the previous question.

(4) Show that there are natural "inclusions"

$$\mathrm{PSL}(2,\mathbb{R}) \hookrightarrow \mathrm{PGL}(2,\mathbb{R}) \hookrightarrow \mathrm{PGL}(2,\mathbb{C})$$
$$\mathrm{PSL}(2,\mathbb{R}) \hookrightarrow \mathrm{PSL}(2,\mathbb{C}) \xrightarrow{\sim} \mathrm{PGL}(2,\mathbb{C})$$

How would you describe the difference between $\mathrm{PSL}(2,\mathbb{R})$ and $\mathrm{PGL}(2,\mathbb{R})$?

Exercise 12. The one-dimensional case

Throughout the chapter, we discussed conformal maps and Möbius transformations of $\widehat{\mathbb{R}^n}$, S^n, H^n, B^n for $n \geqslant 2$. What about the case $n = 1$? Work out as many details as possible about what still works and what breaks.