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Hyperbolic Geometry

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Exercise Sheet 2 (Chapters 3 and 4)

Chapter 3

Exercise 1. Characterization of orthogonal decompositions

Let (V, φ) be a finite-dimensional vector space equipped with a symmetric bilinear form. Let $W \subseteq V$ be a subspace.

(1) Is $\dim W + \dim W^\perp \geqslant \dim V$ always true? Is $W + W^\perp = V$ always true?

(2) Recall the proof that $V = W \oplus W^\perp$ if and only if $\varphi_{|W}$ is nondegenerate.

Exercise 2. Orthogonal subspace to a timelike vector

Prove Proposition 3.8 (copied below) directly, without using the results of § 3.1.

Proposition. *Let V be a Minkowski space. If $v \in V$ is timelike, then $v^\perp$ is a spacelike hyperplane, and $V = \mathbb{R}v \oplus v^\perp$.*

Exercise 3. Time orientation-preserving criterion

Let M be a matrix in $\mathrm{O}(n, 1)$. Show that f is time orientation-preserving if and only if the bottom-right coefficient of M is positive.

Exercise 4. Lorentz boosts and structure of the Lorentz group

(1) Show that any element of $\mathrm{SO}^+(1, 1)$ can uniquely be written:

$$\begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}$$

with $t \in \mathbb{R}$. Show that $\mathrm{SO}^+(1, 1)$ is connected.

(2) An element $f \in \mathrm{SO}^+(n, 1)$ is called a *Lorentz boost* if the set of fixed points of f contains a spacelike subspace of codimension 2. Show that in a suitable basis, a Lorentz boost looks like:

$$\begin{bmatrix} \boxed{I_{n-1}} & 0 \\ 0 & \boxed{\begin{matrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{matrix}} \end{bmatrix}$$

Argue that any Lorentz boost is in the connected component of the identity in $\mathrm{O}(n, 1)$.

(3) Show that for any two unit timelike vectors u and v, there exists a unique Lorentz boost f such that $f(u) = v$.

(4) Show that any matrix $M \in O^+(n,1)$ can uniquely be written as $M = QB$, where B is the matrix of a Lorentz boost and Q is a matrix of the form

$$\begin{bmatrix} \boxed{Q_1} & 0 \\ 0 & 1 \end{bmatrix}$$

with $Q_1 \in \mathrm{O}(n)$.

(5) Recall why $\mathrm{SO}(n)$ is connected (optional) and conclude that $\mathrm{SO}^+(n,1)$ is connected.

Exercise 5. Connected components of the Lorentz group and projective Lorentz group

We recall that the subgroup $\mathrm{SO}^+(n,1) \subseteq \mathrm{O}(n,1)$ is connected (see Exercise 4).

(1) Show that $\mathrm{SO}^+(n,1)$ is the identity component of $\mathrm{O}(n,1)$. Show that it is a normal subgroup. Show that the quotient $\mathrm{O}(n,1)/\mathrm{SO}^+(n,1)$ is isomorphic to the Klein four-group.

(2) Show that the the center of $\mathrm{O}(n,1)$ is equal to the subgroup of homotheties (scalar multiples of the identity) in $\mathrm{O}(n,1)$, that is, $Z(\mathrm{O}(n,1)) = \{\pm I_{n+1}\}$.

(3) Let $\mathrm{PO}(n,1) := \mathrm{O}(n,1)/Z(\mathrm{O}(n,1)) = \mathrm{O}(n,1)/\{\pm I_{n+1}\}$ denote the projective Lorentz group. Can you identify it to a subgroup of $\mathrm{O}(n,1)$?

Chapter 4

Exercise 1. Isometries of the hyperboloid

The goal of this exercise is to dermine the group of isometries of hyperbolic space in the hyperboloid model, in particular to provide a careful proof of Theorem 4.7.
Let $M = \mathbb{R}^{n,1}$ be Minkowski space, denote $\mathcal{H}$ the hyperboloid of two sheets $\mathcal{H} = \{v \in M : \langle v, v \rangle = -1\}$, and $\mathcal{H}^+$ the upper sheet (with $x_{n+1} > 0$.)

(1) The goal of this question is to show that $\mathrm{O}^+(n,1)$ acts by isometries on $\mathcal{H}^+$.
  (a) Show that the action of $\mathrm{O}(n,1)$ on M leaves $\mathcal{H}$ invariant.
  (b) Show that $\mathrm{O}(n,1)$ acts on $\mathcal{H}$ by Riemannian isometries.
  (c) Show that $f \in \mathrm{O}(n,1)$ preserves $\mathcal{H}^+$ if and only if $f \in \mathrm{O}^+(n,1)$. Conclude that $\mathrm{O}^+(n,1) \subseteq \mathrm{Isom}(\mathcal{H}^+)$.
  (d) Optional: Show that $f \in \mathrm{O}^+(n,1)$ is orientation-preserving on $\mathcal{H}^+$ if and only if $f \in \mathrm{SO}^+(n,1)$. Conclude that $\mathrm{SO}^+(n,1) \subseteq \mathrm{Isom}^+(\mathcal{H}^+)$.

(2) The goal of this question is to show that, conversely, any isometry of $\mathcal{H}^+$ is induced by some element of $\mathrm{O}^+(n,1)$ acting on M.
  (a) Show that the action of $\mathrm{O}^+(n,1)$ on $\mathcal{H}^+$ is transitive. *Hint: use Exercise 4 (3).*
  (b) Derive from the previous question that it is enough to show that any isometry of $\mathcal{H}^+$ fixing some point is induced by some element of $\mathrm{O}(n,1)$ acting on M fixing that point.
  (c) Identify the subgroup K of $\mathrm{O}(n,1)$ fixing the point $v_0 = (0,\ldots,0,1)$. Show that the induced action of K in $\mathrm{T}_{v_0}\mathcal{H}^+$ is transitive on the set of orthonormal bases of $\mathrm{T}_{v_0}\mathcal{H}^+$.
  (d) Let f be an isometry of $\mathcal{H}^+$ fixing v_0. Show that f is completely determined by its derivative at v_0.
  (e) Conclude that $\mathrm{Isom}(\mathcal{H}^+) = \mathrm{O}^+(n,1)$ and $\mathrm{Isom}^+(\mathcal{H}^+) = \mathrm{SO}^+(n,1)$.

Exercise 2. Distance between geodesics on the hyperboloid

We denote as usual $\mathcal{H}^+ \subseteq \mathbb{R}^{n,1}$ the upper sheet of the hyperboloid in Minkowski space. Let $p \in \mathcal{H}^+$ and let $v, w \in \mathrm{T}_p \mathcal{H}^+$ be an orthonormal pair of tangent vectors. It is a general fact of Riemannian geometry that the distance between the geodesics $\gamma_v(t)$ and $\gamma_w(t)$ satisfies

$$d\left(\gamma_v(t), \gamma_w(t)\right)^2 = 2t^2 - \frac{1}{3} K\, t^4 + O(t^5)$$

as $t \to 0$, where K denotes the sectional curvature of the plane spanned by v and w. (See § 2.3.3 for more information.)

(1) Show that $d\left(\gamma_v(t), \gamma_w(t)\right) = \operatorname{arcosh}\left(\cosh^2 t\right)$.

(2) Find the Taylor expansion of $\operatorname{arcosh}(\cosh^2 x)$ to order 3 as $x \to 0$.

(3) Conclude that $K = -1$.

(4) Show likewise that $\mathcal{H}^+_R$ has constant sectional curvature $-\frac{1}{R^2}$.

Exercise 3. Jacobi fields on the hyperboloid

We denote as usual $\mathcal{H}^+ \subseteq \mathbb{R}^{n,1}$ the upper sheet of the hyperboloid in Minkowski space.

(1) Let $v, w \in \mathrm{T}_p \mathcal{H}^+$ be an orthonormal pair. Let us define $\gamma \colon \mathbb{R} \times \mathbb{R} \to \mathcal{H}^+$ by

$$\gamma(s,t) = \cosh(t) p + \sinh(t) \left[\cos(s) v + \sin(s) w\right] .$$

Show that:
  (i) $\gamma(s, \cdot)$ is a unit geodesic for all $s \in \mathbb{R}$,
  (ii) $\gamma(0, \cdot) = \gamma_v$.
Such a family γ is called a *variation of geodesics*.

(2) Let $J(t) = \frac{\partial}{\partial s}_{|s=0} \gamma(s,t)$. Check that $J(0) = 0$ and $J'(0) = w$. This is a *normal Jacobi field.*

(3) We admit the following fact: if $J(t)$ is a normal Jacobi field along a unit geodesic and satisfies $J''(t) + k(t) J(t) = 0$, then the sectional curvature of the plane spanned by $\gamma'(t)$ and $J(t)$ is equal to $k(t)$ for all $t > 0$[1]. Show that the plane spanned by v and w has curvature -1.

(4) Conclude that $\mathcal{H}^+$ has constant sectional curvature -1.

(5) Show similarly that the hyperboloid of radius R has constant sectional curvature $-\frac{1}{R^2}$.

Exercise 4. Horocycles on the hyperboloid

Let P be an affine plane in Minkowski space $\mathbb{R}^{2,1}$ whose underlying vector space $\vec{P}$ is the orthogonal of an isotropic vector n. The curve $\mathcal{H}^+ \cap P$ is called a *horocycle*.

(1) Show that $P = \{p \in \mathbb{R}^{2,1} : \langle p, n\rangle = c\}$ where c is a constant.

(2) Optional: Show that any two horocycles are congruent.

(3) Show that any horocycle is a parabola in $\mathbb{R}^{2,1}$.

(4) (*) Show that all the geodesics in $\mathcal{H}^+$ perpendicular to a given horocycle are asymptotic.

[1] Students who know Riemannian geometry should recall why this is true. It follows from the Jacobi equation $J''(t) + R(J(t), \gamma'(t))\gamma'(t) = 0$.

Exercise 5. Comparing hyperboloids

We denote $(\mathcal{H}_R^+, g_R)$ the upper sheet of the hyperboloid of radius R in $\mathbb{R}^{n,1}$ equipped with its Riemannian metric,

(1) Find a natural map $f\colon \mathcal{H}_R^+ \to \mathcal{H}_1^+$.

(2) Compare g_R and $f^* g_1$. Recover the results of § 4.7.

Exercise 6. Euclid's fifth postulate for the hyperboloid

Does Euclid's fifth postulate hold for the hyperboloid model? Compute the angle of parallelism as a function of the distance a (see Figure 1.3).