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Hyperbolic Geometry

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Exercise Sheet 1 (Chapters 1 and 2)

Chapter 1

Exercise 1. Beltrami-Klein disk and Poincaré disk

(1) Prove that Euclid's postulate (E1) holds in the Beltrami-Klein disk.

For now, we cannot really discuss postulates (E2), (E3), and (E4), because we have yet to define distances, angles, and isometries in this model, but we will see that they also hold.

(2) Show that Euclid's postulate (E5) does not hold in the Beltrami-Klein disk.

(3) Repeat the exercise with the Poincaré disk.

Exercise 2. Triangles in the Poincaré disk

We recall that the Poincaré disk model is *conformal*: the angles between two lines (or curves) from the point of view of hyperbolic geometry is the same as their Euclidean angle (i.e., the angle between the tangents).

(1) Draw a right-angled triangle in the Poincaré disk.

(2) Show that in the Poincaré disk, the sum of angles in a triangle is always less than π. Argue that over all nondegenerate triangles, the sum ranges in the interval $(0, \pi)$.

Exercise 3. Independence of Euclid's fifth postulate

Using Gödel's theorem, explain carefully why Beltrami's models for the hyperbolic plane show that hyperbolic geometry is no less consistent than Euclidean geometry. Conclude that if Euclidean geometry is consistent, then Euclid's fifth postulate is independent from the first four.

Remark: This exercise, just like the presentation of § 1.3, is naive: it implicitly assumes that Euclid's system meets the requirements of a theory as defined by first-order logic, where Gödel's theorem applies. This is not quite the case.

Chapter 2

Exercise 1. The sphere

Let S^n_R denote the sphere of radius $R > 0$ centered at the origin in $\mathbb{R}^{n+1}$. We would like to understand geodesics and curvature in S^n_R. This exercise may seem basic, but it is very important: we will follow the same strategy for the hyperboloid in Minkowsi space.

(1) Show that any linear isometry of $\mathbb{R}^{n+1}$ induces a Riemannian isometry of S^n_R. Optional: show that the group of isometries of S^n_R is $\mathrm{O}(n+1)$.

(2) For now, we consider the sphere $S = S^2_R$ in $\mathbb{R}^3$.
   (a) Show that for any $p \in S$ and $v \in \mathrm{T}_p S$, there exists a plane $H \subseteq \mathbb{R}^3$ such that the reflection s_H through H leaves p and v invariant.
   (b) Show that geodesics on S are exactly the great circles (intersection of S with planes through the origin), parametrized with constant speed.
   (c) Show that we have the explicit expression:

$$\gamma_v(t) = \cos(\|v\| t)\, p + R \sin(\|v\| t)\, \frac{v}{\|v\|} \,.$$

   (d) Let $p, q \in S$. Show that their distance on S is given by $d(p,q) = R \angle(p,q)$ where $\angle(p,q)$ denotes the unoriented angle between p and q seen as vectors in $\mathbb{R}^3$.

(3) What is the exterior unit normal N at p? Show that the extrinsic curvature $\rho_p(v)$ is equal to $-\frac{1}{R}$ for any unit vector v. Conclude that the Gaussian curvature is $\frac{1}{R^2}$ at p, and hence everywhere.

(4) Let $n \geqslant 2$.
   (a) Show that (2) remains true with S^n_R instead of S and $\mathbb{R}^{n+1}$ instead of $\mathbb{R}^3$, as long as by *plane* we mean a 2-dimensional subspace.
   (b) Let $P \subseteq \mathrm{T}_p S^n_R$ be a 2-plane. Denote $E_P \subseteq \mathbb{R}^{n+1}$ the subspace spanned by p and P. Show that the union of geodesics in S^n_R with initial velocity in P is the sphere S_P of radius R in E_P. *In the terminology of Riemannian geometry:* $\exp_p(P) = S_P$.
   (c) Conclude that S^n_R has constant sectional curvature $\frac{1}{R^2}$.

Exercise 2. The tractricoid

One of the obstacles to the discovery of the hyperbolic plane is that it cannot be smoothly completely embedded as a surface in $\mathbb{R}^3$. [1] However, it is possible to smoothly embed a piece of the hyperbolic plane in $\mathbb{R}^3$, as this exercise illustrates.

(1) Consider the *tractrix* curve in the xz-plane parametrized by:

$$\begin{aligned} \gamma\colon [0, +\infty) &\longrightarrow \mathbb{R}^3 \\ t &\longmapsto (x(t) = \operatorname{sech} t, y(t) = 0, z(t) = t - \tanh t) \end{aligned}$$

where $\operatorname{sech} = \frac{1}{\cosh}$ is the hyperbolic secant and $\tanh = \frac{\sinh}{\cosh}$ is the hyperbolic tangent. Draw the tractrix in the plane. Optiona: Show that the tractrix is the path followed by a reluctant dog on a leash (in German, a tractrix is a *Hundekurve*).

(2) The *tractricoid* (sometimes called *pseudosphere*[2]) is the surface S in $\mathbb{R}^3$ obtained by rotating

[1]There are no complete surfaces of constant Gaussian curvature -1 of class C^2 in $\mathbb{R}^3$ (Efimov's theorem, 1964 [**?**], also see [**?**]). Hilbert first proved it for class C^4 in 1901. Surprisingly, there are C^1 embeddings of the hyperbolic plane in $\mathbb{R}^3$. This is a corollary of the Nash-Kuiper C^1 embedding theorem. See http://www.math.cornell.edu/~dwh/papers/crochet/crochet.html for illustrations of crocheted hyperbolic planes.

[2]Depending on authors, *pseudosphere* may refer to the tractricoid specifically, or to any surface in $\mathbb{R}^3$ of Gaussian curvature -1. I find the term more appropriate for level sets of the quadratic form in a pseudo-Euclidean vector space (this includes the hyperboloid model of the hyperbolic plane).

the tractrix defined above around the z-axis. Show that it has parametric equations:

$$\begin{aligned} x &= \operatorname{sech} t \cos\theta \\ y &= \operatorname{sech} t \sin\theta \\ z &= t - \tanh t\,. \end{aligned}$$

Show that rotations around the z-axis and reflections through vertical planes containing the z-axis are isometries of S. Draw a sketch of S.

(3) We denote $f(\theta,t) := (x(\theta,t), y(\theta,t), z(\theta,t))$. Consider the curves $c_t(\theta) = f(\theta,t)$ when t is fixed ("parallels") and $\gamma_\theta(t) = f(\theta,t)$ when θ is fixed ("meridians"). Draw such curves on S. Using a symmetry argument, show that the curves $\gamma_\theta(t)$ are geodesics up to parametrization.

(4) Consider a point $p = f(\theta_0, t_0)$ on the tractricoid. Our goal is to show that the Gaussian curvature of S at p is -1.
   (a) Explain why it is enough to show it when $\theta_0 = 0$.
   (b) Compute the velocities of c_{t_0} and γ_0 at p. Derive an expression of the unit normal vectors at p.
   (c) Compute the extrinsic curvatures of S at p in the unit directions tangent to c_{t_0} and γ_0.
   (d) Using a symmetry argument, explain why the principal directions of curvatures of S at p must be tangent to c_{t_0} or γ_0. Derive the value of the principal curvatures at p, conclude that S has Gaussian curvature -1 at p, and hence everywhere.

(5) Compute the arclength parameter of $\gamma(t)$. Show that the tractricoid is incomplete.

Exercise 3. The Poincaré disk

The *Poincaré disk* $\mathbb{D}$ is defined as the unit disk equipped with the Riemannian metric:

$$\mathrm{d}s^2 = \frac{4\left(\mathrm{d}x^2 + \mathrm{d}y^2\right)}{(1 - x^2 - y^2)^2}$$

In this exercise, we denote $O \in \mathbb{D}$ the point which is at the origin in $\mathbb{R}^2$.

(1) Show that the Poincaré metric on $\mathbb{D}$ is conformal to the Euclidean metric. Is the Euclidean metric complete on $\mathbb{D}$?

(2) Show that any $f \in \mathrm{O}(2)$ induces an isometry of $\mathbb{D}$ that fixes O. Optional: show the converse.

(3) Show that any diameter of $\mathbb{D}$ (straight chord through the origin) is a geodesic. *Hint: consider the fixed points of a reflection $f \in \mathrm{O}(2)$.*

(4) Find a parametrization of geodesics through the origin. Find an expression of the distance between O and an arbitrary point in $\mathbb{D}$.

(5) Show that $\mathbb{D}$ is complete. *Use the Hopf-Rinow theorem.*

(6) Compute the curvature of $\mathbb{D}$.

Exercise 4. Euclid's postulates for Riemannian surfaces (*)

Give an interpretation of Euclid's postulates for Riemannian surfaces and discuss their implications.

This exercise is not easy, and probably only suitable for students with a strong background of Riemannian geometry. Regardless, I recommend that you read the solution.