Brice Loustau
TU Darmstadt

General Relativity

Summer 2019

Exercise Sheet 6

Exercise 1. FLRW spacetimes: vacuum case

Consider a FLRW spacetime $M = I \times \Sigma_\kappa$ with scale factor $a(t)$: the metric is $g = -\mathrm{d}t^2 + a(t)g_\kappa$ where (Σ_k, g_κ) is the space form of constant sectional curvature $k \in \mathbb{R}$. We recall that the extended Einstein tensor $G = \mathrm{Ric} - \frac{1}{2}Sg + \Lambda g$ is given by

$$G(U,U) = 3\frac{\kappa + (a')^2}{a^2} - \Lambda \qquad G\left(\frac{1}{a}X, \frac{1}{a}X\right) = -\frac{\kappa + (a')^2}{a^2} - 2\frac{a''}{a} + \Lambda$$

where $U = \partial_t$ and X is any unit tangent vector to Σ_k.

By convention in this exercise the time $t = 0$ represents the present time.

(1) Show that the Einstein equations in the vacuum are equivalent to

$$\frac{a''}{a} = \frac{\Lambda}{3} = \kappa + H_0^2 \,.$$

We recall that $H = \frac{a'}{a}$ is the Hubble parameter and we denote $H_0 := H(0)$.
Argue that given $a(0) = 1$ and $H_0 \in \mathbb{R}$, there exists a unique solution $a(t)$ defined on some maximal interval I.

(2) Assume $\Lambda = 0$.

(a) Show that if $H_0 = 0$, then $I = \mathbb{R}$ and M is Minkowski spacetime.

(b) Discuss the case $H_0 > 0$ qualitatively: Show that the universe keeps expanding. Is the universe finite? Show that there was a Big Bang. Was the whole universe reduced to a point then? Will there be a Big Crunch in the distant future?

(c) Discuss the case $H_0 < 0$ qualitatively.

(3) Assume $\Lambda > 0$.

(a) Find the general solution $a(t)$ with initial conditions as in (1).

(b) Sketch the graph of the function $t \mapsto a(t)$ depending on the value of H_0: Distinguish cases $H_0 < -1$, $H_0 = -1$, $-1 < H_0 < 0$, $H_0 = 0$, $0 < H_0 < 1$, $H_0 > 1$.

(c) Discuss each case qualitatively.

(4) Assume $\Lambda < 0$.

(a) Find the general solution $a(t)$ with initial conditions as in (1). Check that it can be written: $a(t) = A\cos(\lambda t - t_0)$ where $\lambda = \sqrt{-\frac{\Lambda}{3}}$, $A = \sqrt{1 + H_0^2}$ and $t_1 = \arctan(H_0)$.

(b) Sketch the graph of the function $t \mapsto a(t)$ depending on the sign of H_0 and discuss each case qualitatively.