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General Relativity

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Exercise Sheet 5

Exercise 1. A surface of revolution

We recall that the Schwarzschild solution on $M = \mathbb{R} \times (2m, +\infty) \times S^2$ is given by:

$$g = -\left(1 - \frac{2m}{r}\right) \mathrm{d}t^2 + \left(1 - \frac{2m}{r}\right)^{-1} \mathrm{d}r^2 + r^2\, \mathrm{d}\sigma^2$$

where $\mathrm{d}\sigma^2$ is the line element (i.e. metric) on S^2, which is given in spherical coordinates (θ, φ) by $\mathrm{d}\sigma^2 = \mathrm{d}\theta^2 + \sin^2\theta\, \mathrm{d}\varphi^2$.

(1) Show that the subset $S \subseteq M$ defined by $\{t = t_0, \theta = \pi/2\}$ (where t_0 is a constant) is an embedded surface in M. Write down the induced metric on S. Is it Lorentzian?

(2) Find a function $z = z(r)$ such that the map $f(r, \varphi) = (r\cos\varphi, r\sin\varphi, z(r))$ defines an isometric embedding from S to $\mathbb{R}^3$. Draw a picture of the embedded surface in $\mathbb{R}^3$.

Exercise 2. Schwarzschild solution with cosmological constant

For this exercise we use the setting, notations, and results of §3.1 of Prof. KGB's course notes: *Spherically symmetric static Lorentz manifolds*.

(1) Let us call $\{\lambda_i\}$ the eigenvalues of the Ricci tensor, more precisely, λ_1, λ_2, $\lambda_3 = \lambda_4 \in \mathbb{R}$ are defined by $\mathrm{Ric}(\partial_t, \partial_t) = \lambda_1 g(\partial_t, \partial_t)$, $\mathrm{Ric}(\partial_\rho, \partial_\rho) = \lambda_2 g(\partial_\rho, \partial_\rho)$, and $\mathrm{Ric}(X, X) = \lambda_3 g(X, X)$ for any tangent vector X along the S^2 factor. Recall why such λ_i's exist and why their values are:

$$\lambda_1 = -\frac{a''}{a} - 2\frac{a'b'}{ab}$$
$$\lambda_2 = -\frac{a''}{a} - 2\frac{b''}{b}$$
$$\lambda_3 = \lambda_4 = \frac{1}{b^2} - \frac{b'^2}{b^2} - \frac{b''}{b} - \frac{a'b'}{ab}$$

(2) Argue that Einstein's equations in the vacuum with cosmological constant Λ are equivalent to $\lambda_1 = \lambda_2 = \lambda_3 = \Lambda$.

(3) Show that the equation $\lambda_1 = \lambda_2$ leads to $a = C\, b'$ where C is a constant, and one can assume $C = 1$ after rescaling the t-coordinate.

(4) Show that the equations $\lambda_3 = \lambda_2 = \Lambda$ lead to $\frac{b'''}{b'} + 2\frac{b''}{b} = -\Lambda$.

(5) Show that $m := b^2 b'' + \frac{\Lambda}{3} b^3$ is constant. Show that b satisfies the first order ODE

$$b'^2 = 1 - \frac{2m}{b} - \frac{\Lambda}{3} b^2 \,.$$

(6) Show that after the change of coordinates $r = b$, the Schwarzschild metric is:

$$g = -\left(1 - \frac{2m}{r} - \frac{\Lambda}{3} r^2\right) \mathrm{d}t^2 + \left(1 - \frac{2m}{r} - \frac{\Lambda}{3} r^2\right)^{-1} \mathrm{d}r^2 + r^2\, \mathrm{d}\sigma^2$$

Exercise 3. Sectional curvatures of the Schwarzschild metric

Consider the manifold $M = \mathbb{R} \times (0, +\infty) \times S^2 \approx \mathbb{R} \times (\mathbb{R}^3 - \{0\})$ with coordinates (t, r, p). We recall that the Schwarzschild metric g is defined on the open set $U \subseteq M$ defined by $\{r > 2m\}$.

(1) Show that the Schwarzschild metric cannot be extended to a smooth Riemannian metric in M.

(2) Compute (or recover from the course notes) all the sectional curvatures of the Schwarzschild metric (with or without cosmological constant). Show that the sectional curvatures remain bounded when $r \to 2m$.

(3) Show that there exists a proper isometric analytic embedding of (U, g) is a Lorentzian manifold N. *Think of the Kruskal-Szekeres spacetime.*

(4) (*) In what sense is the singularity of the Schwarzschild solution at $r = 2m$ an "apparent singularity"? Come up with a convincing mathematical and physical explanation in your own words.

Exercise 4. Falling straight into a blackhole

The goal of this exercise is to show that it takes finite proper time for a particle falling straight into a blackhole to reach the event horizon.

Consider the Schwarzschild metric in the (t, ρ, p)-coordinates where $(t, \rho, p) \in \mathbb{R} \times (\rho_0, +\infty) \times S^2$. We refer to the course notes for more details. Consider a curve $c(s) = (t(s), \rho(s), p(s) = p_0)$ where p_0 is some fixed point on S^2 with $\theta(p_0) = \pi/2$.

(1) Show that the geodesic equations for such a curve are:

$$\ddot{t} + 2\frac{b''(\rho)}{b'(\rho)}\dot{t}\dot{\rho} = 0$$
$$\ddot{\rho} + b'(\rho)b''(\rho)\dot{t}^2 = 0$$
$$\ddot{\theta} = 0$$
$$\ddot{\varphi} = 0$$

Show that there such geodesics exist and are determined by their initial condition $c(0) = (t(0), \rho(0), p_0)$ and $\dot{c}(0) = (\dot{t}(0), \dot{\rho}(0), 0)$.

Consider the maximal interval of existence $[0, \beta)$ of such a geodesic c where $\beta \in [0, +\infty]$. The goal of the exercise is to show that $\beta < +\infty$ and $\lim_{s \to \beta} \rho(s) = \rho_0$. (We recall that ρ_0 is characterized by $b'(\rho_0) = 0$).

(2) From now on we assume that c as before is a geodesic with $\dot{\rho}(0) < 0$. Show that $\ddot{\rho}(s) \leqslant 0$ and $\dot{\rho}(s) \leqslant \dot{\rho}(0) < 0$ for all $s \geqslant 0$. Conclude that $\beta < +\infty$ (hint: don't forget that ρ must stay $> \rho_0$ for all s). Extend the argument to the case $\dot{\rho}(0) = 0$.

(3) It remains to show that $\lim_{s \to \beta} \rho(s) = \rho_0$. By contradiction, assume not. Explain why it is enough to argue that $t(s)$ remains bounded when $s \to \beta$ in order to reach a contradiction. First observe that $b'(\rho(s))$ remains bounded. Then derive from the first geodesic equation that $\dot{t}(s)b'(\rho(s))^2$ is constant. (Is this expected? Recall the energy of a Schwarzschild geodesic!). Conclude.

(4) (*) *Optional.* Assume additionally that c is timelike. Show that $\lim_{s \to \beta} t(s) = +\infty$. *Hint: start by deriving from the fact that $\dot{t}(s)b'(\rho(s))^2$ is constant that $\dot{t}(s) = O((\rho(s) - \rho_0(s))^{-2})$ when $s \to \beta$.*

(5) Is the Schwarzschild metric complete?

Exercise 5. Energy and angular momentum for Schwarzschild geodesics, Noether's theorem

(1) Recall the definition of the energy E and the angular momentum L of a Schwarzschild geodesic.

(2) Noether's theorem states that for any Killing vector field X and for any geodesic γ in a semi-Riemannian manifold (M, g), the quantity $g(\gamma', X)$ is constant along the geodesic.

  (a) Prove Noether's theorem.

  (b) Explain why in a physical setting, symmetries for the Lagrangian (e.g. the integrand of the Hilbert-Einstein functional) will typically produce Killing vector fields, and therefore invariant quantities along geodesics.

  (c) Show that the energy E and the angular momentum L of a Schwarzschild geodesic are constant as an application of Noether's theorem.

Exercise 6. (*) Tidal forces and spaghettification

(1) Read the Wikipedia article *Spaghettification*:

https://en.wikipedia.org/wiki/Spaghettification

(2) Explain the phenomenon of spaghettification with the following mathematical setup. Consider two particles $c_0(s)$ and $c_1(s)$ falling straight into a blackhole as in Exercise 4. These particles represent, say, the two extremities of a rope (or a human body). The goal is to understand the difference in relative acceleration between $c_0(s)$ and $c_1(s)$ with a qualitative argument. In between these two geodesics, there is a family of geodesics $c_\theta(s)$ for $\theta \in [0, 1]$. Therefore we get a Jacobi field...