Quiz #8 Solutions

Problem 1.

Consider the ring $R = \mathbb{Z}/5\mathbb{Z}$.

(1)

+	[0]	[1]	[2]	[3]	[4]
[0]	[0]	[1]	[2]	[3]	[4]
[1]	[1]	[2]	[3]	[4]	[0]
[2]	[2]	[3]	[4]	[0]	[1]
[3]	[3]	[4]	[0]	[1]	[2]
[4]	[4]	[0]	[1]	[2]	[3]

(2)

·	[0]	[1]	[2]	[3]	[4]
[0]	[0]	[0]	[0]	[0]	[0]
[1]	[0]	[1]	[2]	[3]	[4]
[2]	[0]	[2]	[4]	[1]	[3]
[3]	[0]	[3]	[1]	[4]	[2]
[4]	[0]	[4]	[3]	[2]	[1]

(3) We can read from the multiplication table that:

$$\begin{aligned}[1]\cdot[1] &= [1]\\ [2]\cdot[3] &= [1]\\ [3]\cdot[2] &= [1]\\ [4]\cdot[4] &= [1]\,.\end{aligned}$$

This shows that [1], [2], [3], and [4] all admit an inverse, namely, [1], [3], [2], and [4] respectively.

The ring $\mathbb{Z}/5\mathbb{Z}$ is commutative, has more than one element, and satisfies the property that any nonzero element has an inverse, therefore it is a **field**.

Problem 2.

(1) We could show that C is a ring directly by checking all the requirements. Instead, let us show that C is a subring of $\mathcal{M}_2(\mathbb{R})$: this method is somewhat faster. In general, in order to show that $S \subset R$ is a subring, it is enough to show that it satisfies the *subring test*:
- $1_R \in S$
- $\forall (x, y) \in S^2 \quad x - y \in S$
- $\forall (x, y) \in S^2 \quad x \cdot y \in S$

In the present situation, with $R = \mathcal{M}_2(\mathbb{R})$ and $S = C$:
- $1_R \in S$ is true (take $a = 1$ and $b = 0$).
- $\forall (x, y) \in S^2 \quad x - y \in S$ is true: if $x = \begin{bmatrix} a_1 & -b_1 \\ b_1 & a_1 \end{bmatrix}$ and $y = \begin{bmatrix} a_2 & -b_2 \\ b_2 & a_2 \end{bmatrix}$, then $x + y = \begin{bmatrix} a_1 + a_2 & -(b_1 + b_2) \\ b_1 + b_2 & a_1 + a_2 \end{bmatrix}$ is an element of C (take $a = a_1 + a_2$ and $b = b_1 + b_2$).
- $\forall (x, y) \in S^2 \quad x - y \in S$ is true: if $x = \begin{bmatrix} a_1 & -b_1 \\ b_1 & a_1 \end{bmatrix}$ and $y = \begin{bmatrix} a_2 & -b_2 \\ b_2 & a_2 \end{bmatrix}$, then $x \cdot y = \begin{bmatrix} a_1 + a_2 - b_1 + b_2 & -(a_1b_2 + a_2b_1) \\ a_1b_2 + a_2b_1 & a_1 + a_2 - b_1 + b_2 \end{bmatrix}$ is an element of C (take $a = a_1 + a_2 - b_1 + b_2$ and $b = a_1b_2 + a_2b_1$).

(2) Similar computations to the one that we did above show that φ satisfies all the requirements of a ring homomorphism, namely:
- $\varphi(1) = 1_R$.
- $\forall (z_1, z_2) \in \mathbb{C}^2 \quad \varphi(z_1 + z_2) = \varphi(z_1) + \varphi(z_2)$.
- $\forall (z_1, z_2) \in \mathbb{C}^2 \quad \varphi(z_1 \cdot z_2) = \varphi(z_1) \cdot \varphi(z_2)$.

(3) We notice that $M = \varphi(z)$ where $z = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2} = e^{i\pi/4}$. Since φ is a homomorphism, it follows that $M^n = \varphi(z)^n = \varphi(z^n)$ for any integer n. In particular, for $n = 100$, we find that $M = \varphi(z^{100}) = \varphi(1)$, so that:

$$M^{100} = 1_C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\,.$$