Quiz #7 Solutions

Problem 1.

First of all, we already know that $\operatorname{Ker}\varphi$ is a subgroup of G. In order to show that it is a normal subgroup, we need to show that, for every $x \in \operatorname{Ker}\varphi$ and for every $g \in G$, $gxg^{-1} \in \operatorname{Ker}\varphi$. In other words we need to show that $\varphi(gxg^{-1}) = e'$, where e' denotes the identity element of G'. This is a straightforward computation, using the properties of a group homomorphism:

$$\begin{aligned}\varphi(gxg^{-1}) &= \varphi(g)\varphi(x)\varphi(g^{-1})\\ &= \varphi(g)\varphi(x)\varphi(g)^{-1}\\ &= \varphi(g)e'\varphi(g)^{-1}\\ &= \varphi(g)\varphi(g)^{-1}\\ &= e'\end{aligned}$$

Problem 2.

Consider the map

$$\begin{aligned}\varphi\colon G &\longrightarrow G'\\ k &\longmapsto e^{2ik\pi/n}\end{aligned}$$

where $G = \mathbb{Z}$ and $G' = U_n$.

This map, which we have seen several times in class, is a group homomorphism. Moreover, its kernel is $n\mathbb{Z}$. By the first isomorphism theorem, we get $G/\operatorname{Ker}\varphi \approx \operatorname{Im}\varphi$, in other words $\mathbb{Z}/n\mathbb{Z} \approx U_n$.

Problem 3.

(1) $x + y = [3] + [4] = [7] = [1]$.

(2) $-x = -[3] = [-3] = [3]$ and $-y = -[4] = [-4] = 2$.

(3) $x = [3]$ and $2x = [6] = [0]$, therefore x is of order 2.
$y = [4]$, $2y = [8] = [2]$, $3y = [12] = [0]$, therefore y is of order 3.