Quiz #6 Solutions

Monday, November 6 2017

Problem 1.

(1)

$$\sigma^2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 4 & 3 & 5 & 2 \end{pmatrix}$$
$$\sigma^3 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 2 & 1 & 4 & 5 \end{pmatrix}$$
$$\sigma^4 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 5 & 3 & 2 & 4 \end{pmatrix}$$
$$\sigma^5 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 1 & 5 & 2 \end{pmatrix}$$
$$\sigma^6 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix}$$

(2) The identity element in the group S_5 is the identical permutation

$$id = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix}.$$

We see from our answer to the previous answer that the smallest positive integer $n \in \mathbb{N}$ such that $\sigma^n = id$ is $n = 6$. Thus, σ has order 6 in S_5.

(3) σ is a generator of S_5 provided that $\langle\sigma\rangle = S_5$. However, we know from the previous question that $\langle\sigma\rangle$ has 6 elements. Indeed: $\langle\sigma\rangle = \{id, \sigma, \ldots, \sigma^5\}$. Therefore we see that $\langle\sigma\rangle \neq S_5$, because S_5 has more than 6 elements: it has $5! = 120$ elements. *Note: S_5 is not a cyclic group, so it cannot be generated by one element.*

(4) σ has two orbits: $\{1, 3\}$ and $\{2, 4, 5\}$.

(5) $\sigma = (1, 3)(2, 5, 4) = (2, 5, 4)(1, 3)$.

(6) $\sigma = (1, 3)(2, 4)(2, 5)$ (there are other possibilities).

(7) Since σ is a product of 3 transpositions, and 3 is an odd number, σ has signature -1. Alternatively: since σ has 2 orbits, its discriminant is $disc(\sigma) = 5 - 2 = 3$, therefore the signature of σ is $(-1)^3 = -1$.

Problem 2. The map

$$\begin{aligned} sign\colon S_n &\longrightarrow \{-1, 1\} \\ \sigma &\longmapsto sign(\sigma) \end{aligned}$$

is well-defined as a map from S_n to $\{-1, 1\}$. In order to show that it is a group homomorphism, we need to show that:

$$\forall \sigma \in S_n \ \forall \tau \in S_n \qquad sign(\sigma\tau) = sign(\sigma)sign(\tau) \ .$$

Let σ and τ be any two elements of S_n; let us prove that $sign(\sigma\tau) = sign(\sigma)sign(\tau)$.

We know that σ and τ can be both be written as a product of transpositions:

$$\begin{aligned} \sigma &= s_1 s_2 \dots s_N \\ \tau &= t_1 t_2 \dots t_M \end{aligned}$$

where N and M are nonnegative integers, and $s_1, s_2, \dots, s_N$ as well as $t_1, t_2, \dots, t_M$ are transpositions.

By definition of the signature,

$$\begin{aligned} sign(\sigma) &= (-1)^N \\ sign(\tau) &= (-1)^M \ . \end{aligned}$$

Taking the product of σ and τ, we see that it can be written as a product of transpositions as:

$$\sigma\tau = s_1 s_2 \dots s_N t_1 t_2 \dots t_M$$

There are $N + M$ transpositions in this product, therefore:

$$\begin{aligned} sign(\sigma\tau) &= (-1)^{N+M} \\ &= (-1)^N \, (-1)^M \\ &= sign(\sigma)sign(\tau) \ . \end{aligned}$$

This was the identity required, therefore we have successfully proved that $sign\colon S_n \rightarrow \{-1, 1\}$ is a group homomorphism.