Quiz #5 Solutions

Monday, October 30 2017

Problem 1.

(1) List of elements in $G = U_8$ in polar form:

$$\begin{aligned} U_8 &= \left\{ e^{i\frac{k\pi}{4}}, 0 \leqslant k \leqslant 7 \right\} \\ &= \left\{ e^{0i}, e^{i\frac{\pi}{4}}, e^{i\frac{\pi}{2}}, e^{i\frac{3\pi}{4}}, e^{i\pi}, e^{i\frac{5\pi}{4}}, e^{i\frac{3\pi}{2}}, e^{i\frac{7\pi}{4}} \right\} \; . \end{aligned}$$

We see indeed that $G = \{\zeta^n, 0 \leqslant \zeta \leqslant 7\}$ where $\zeta = e^{i\frac{\pi}{4}}$.

(2) Yes, G is a cyclic group because $G = \langle \zeta \rangle$.

(3) Subgroup generated by $\zeta^0 = 1$:

$$\langle 1 \rangle = \{1\} \; .$$

Subgroup generated by ζ^1:

$$\begin{aligned} \langle \zeta \rangle &= \{\zeta^0, \zeta^1, \zeta^2, \zeta^3, \zeta^4, \zeta^5, \zeta^6, \zeta^7\} \\ &= G \; . \end{aligned}$$

Subgroup generated by ζ^2:

$$\begin{aligned} \langle \zeta^2 \rangle &= \{\zeta^0 = 1, \zeta^2, \zeta^4, \zeta^6, \zeta^8 = 1\} \\ &= \{1, \zeta^2, \zeta^4, \zeta^6\} \; . \end{aligned}$$

Subgroup generated by ζ^3:

$$\begin{aligned} \langle \zeta^3 \rangle &= \{\zeta^0 = 1, \zeta^3, \zeta^6, \zeta^9 = \zeta, \zeta^{12} = \zeta^4, \zeta^{15} = \zeta^7, \zeta^{18} = \zeta^2, \zeta^{21} = \zeta^5, \zeta^{24} = 1\} \\ &= G \; . \end{aligned}$$

Subgroup generated by ζ^4:

$$\begin{aligned} \langle \zeta^4 \rangle &= \{\zeta^0 = 1, \zeta^4, \zeta^8 = 1\} \\ &= \{1, -1\} \; . \end{aligned}$$

Subgroup generated by ζ^5:

$$\langle\zeta^5\rangle = \{\zeta^0 = 1, \zeta^5, \zeta^{10} = \zeta^3, \zeta^{15} = \zeta^7, \zeta^{20} = \zeta^4, \zeta^{25} = \zeta, \zeta^{30} = \zeta^6, \zeta^{35} = \zeta^3, \zeta^{40} = 1\} \\ = G\,.$$

Subgroup generated by ζ^6:

$$\langle\zeta^6\rangle = \{\zeta^0 = 1, \zeta^6, \zeta^{12} = \zeta^4, \zeta^{18} = \zeta^2, \zeta^{24} = 1\} \\ = \{\zeta^0, \zeta^2, \zeta^4, \zeta^6\}\,.$$

Subgroup generated by ζ^7:

$$\langle\zeta^7\rangle = \{\zeta^0 = 1, \zeta^7, \zeta^{14} = \zeta^6, \zeta^{21} = \zeta^5, \zeta^{28} = \zeta^4, \zeta^{35} = \zeta^3, \zeta^{42} = \zeta^2, \zeta^{49} = \zeta, \zeta^{56} = 1\} \\ = G\,.$$

(4) From the previous answer, we see that the generators of G are ζ, ζ^3, ζ^5 and ζ^7.

(5) We know that any subgroup of a cyclic group is cyclic. Therefore, all the subgroups of G must be in the list in the answer of question 3. Here is the list of subgroups:

$$\{\zeta^0\}$$
$$\{\zeta^0, \zeta^4\}$$
$$\{\zeta^0, \zeta^2, \zeta^4, \zeta^6\}$$
$$G$$

Problem 2.

(1) For any integers a and b, there exists a unique pair of integers (q, r) such that $a = bq + r$ and $0 \leqslant r < |b|$.

(2) For any two integers a and b, their greatest common divisor d is the unique nonnegative integer such that the subgroup of $\mathbb{Z}$ generated by a and b is equal to $d\mathbb{Z}$.

For any two integers a and b, their lowest common multiple m is the unique nonnegative integer such that $a\mathbb{Z} \cap b\mathbb{Z} = m\mathbb{Z}$.

(3) $29 = 4 \times 7 + 1$.

(4) $10\mathbb{Z} \cap 6\mathbb{Z} = 30\mathbb{Z}$.

(5) The subgroup generated by 6 and 9 in $\mathbb{Z}$ is $3\mathbb{Z}$.