Homework

Monday, October 9 2017

Problem 1.

Let S be a nonempty set. Consider the magma $(S^S, \circ)$ where:
- S^S denotes the set of all functions $S \longrightarrow S$.
- $\circ$ denotes the composition of functions.

(1) Show that $(S^S, \circ)$ is a monoid.

(2) Let $f \in S^S$. Show that:
   (a) f has a left inverse if and only if f is surjective.
   (b) f has a right inverse if and only if f is injective.
   (c) f has an inverse if and only if f is bijective.

(3) When f is bijective, its (unique!) inverse is called the *inverse function of* f (and often denoted f^{-1}). Show that $g \in S^S$ is the inverse function of f if and only if:

$$\forall x \in S\ \forall y \in S \quad y = f(x) \iff x = g(y)$$

(4) Do the following functions have an inverse? If so, find it.
   (a)

$$\begin{aligned} id_S \colon S &\longrightarrow S \\ x &\longmapsto x \end{aligned}$$

   (b)

$$\begin{aligned} f \colon S = \{1, 2, 3, 4\} &\longrightarrow S \\ 1 &\longmapsto 2 \\ 2 &\longmapsto 4 \\ 3 &\longmapsto 1 \\ 4 &\longmapsto 3 \end{aligned}$$

   (c)

$$\begin{aligned} g \colon \mathbb{R} &\longrightarrow \mathbb{R} \\ x &\longmapsto x^2 \end{aligned}$$

(d)

$$\begin{aligned} h\colon \mathbb{R} &\to \mathbb{R} \\ x &\mapsto x^3 \end{aligned}$$

(e)

$$\begin{aligned} i\colon \mathbb{R} &\to \mathbb{R} \\ x &\mapsto \tan(x) \end{aligned}$$

(f)

$$\begin{aligned} j\colon \mathbb{R} &\to \mathbb{R} \\ x &\mapsto \frac{1+\tan(2x)}{3} \end{aligned}$$

Problem 2.

Let $(S, *)$ be a monoid.

(1) Can an idempotent element have an inverse?

(2) Let M be a square matrix such that $M^2 = M$ and M is not the identity matrix. Show that $\det M = 0$.

(3) An element $X \in S$ is called an *element of torsion* when there exists an integer $n \geqslant 2$ such that $x^n = e$, where $x^n = x * x * \cdots * x$ (n times) and e is the identity element of S. Show that any element of torsion has an inverse.

Problem 3.

Let $(S, *)$ be a monoid. Let V (resp. W) denote the subset of S consisting of elements which have a left (resp. right) inverse.

(1) Show that V and W are closed under $*$.

(2) What is $V \cap W$? Is it closed?