RUTGERS
UNIVERSITY | NEWARK

Math 451
Abstract Algebra I

Fall 2017

Exam #2

Monday, November 13 2017

Duration: 1H20

NAME: ________________________

Please write clearly and properly. Justify your answers carefully.

Problem	**Grade**
1	
2	
3	
Total	

Problem 1 (~ 8 points).

Let S_{10} denote the symmetric group on 10 letters. Consider the following permutation $\sigma \in S_{10}$:

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 6 & 10 & 8 & 7 & 3 & 5 & 2 & 9 & 1 & 4 \end{pmatrix}$$

(1) What are the orbits of σ?

(2) Find the discriminant of σ and derive the signature of σ.

(3) Write σ as a product of 2 disjoint cycles: $\sigma = C_1 C_2$.

(4) Show that for any $k \in \mathbb{N}$, $\sigma^k = C_1{}^k C_2{}^k$.

(5) Derive from the two previous questions that σ^k is the identity permutation if and only if $k \in m_1\mathbb{Z} \cap m_2\mathbb{Z}$, where m_1 is the length of C_1 and m_2 is the length of C_2.

(6) Derive from the previous question that σ has order 12.

We recall that the order of an element x in a group G is the smallest positive integer $k \in \mathbb{N}$ such that $x^k = e$, where e is the identity element.

(7) Find σ^{18}.

(8) Find σ^{2017}.

Hint: $2017 = 12 \times 168 + 1$.

Problem 2 ($\sim$ 6 points)**.**

Let G be a finite group with identity element e. Denote by N the cardinality of G.

We recall that the order *of an element $x \in G$ is the smallest positive integer $k \in \mathbb{N}$ such that $x^k = e$.*

NB: If you get stuck on a question, you may skip it and still use the result in the next questions.

(1) Show that the order of any element $x \in G$ exists and is a divisor of N.
Hint: Consider the subgroup generated by x and use the theorem of Lagrange.

(2) Show that if there exists an element of order N, then G is a cyclic group.

(3) Show that if N is a prime number, then G is a cyclic group.

(4) Show that if G is a cyclic group, then:

- If N is odd, there exists no element of order 2.
- If N is even, there exists exactly one element of order 2.

(5) Let $G = S_n$ be the symmetric group on n letters. Show that any transposition is an element of order 2. (Bonus question: Are there any other elements of order 2?)

(6) Show that S_n is not a cyclic group unless $n = 2$.

Problem 3 (~ 2 points)**.**

Consider the group $\mathbb{Z} = (\mathbb{Z}, +)$ and the direct product $\mathbb{Z}^2 = \mathbb{Z} \oplus \mathbb{Z}$. Consider the map

$$\begin{aligned} \varphi \colon \mathbb{Z}^2 &\to \mathbb{Z} \\ (x, y) &\mapsto 2x - 3y \ . \end{aligned}$$

Show that φ is a group homomorphism. Bonus question: is φ a group isomorphism?