RUTGERS
UNIVERSITY | NEWARK

Math 451
Abstract Algebra I

Fall 2017

Exam #1

Monday, October 16 2017

Duration: 1H20

NAME: ________________________

Please write clearly and properly. Justify your answers carefully.

Problem	**Grade**
1	
2	
3	
4	
Total	

Problem 1 ($\sim$ 4 points)**.**

Let $n \in \mathbb{N}$ and denote by U_n the set of n-th roots of unity in $\mathbb{C}$. Show that $(U_n, \times)$ is a monoid. Does every element of U_n have an inverse?

Problem 2 ($\sim$ 4 points)**.**

Let $(M, \otimes)$ and $(N, \diamond)$ be two monoids with identity elements e_M and e_N respectively. Let $f\colon M \to N$ be a homomorphism. Denote by $K \subseteq M$ the set of preimages of e_N by f:

$$K = \{x \in M \mid f(x) = e_N\} .$$

Show that K is closed in $(M, \otimes)$.

Problem 3 (~ 7 points)**.**

Let $(M, *)$ be a monoid. Let e denote the identity element of M.

We recall that $x \in M$ is called idempotent when $x * x = x$.
We recall that $x \in M$ is called invertible when there exists an element $y \in M$ which is an inverse of x.

(1) Let $x \in M$. Show that x is idempotent and invertible if and only if $x = e$.

(2) Let $n \in N$ and denote by $\mathcal{M}_n(\mathbb{R})$ the set of all $n \times n$ matrices with real coefficients. Let $M \in \mathcal{M}_n(\mathbb{R})$ such that $M^2 = M$ and $\det(M) \neq 0$. What can you say about M?

(3) Let $M = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} \in \mathcal{M}_2(\mathbb{R})$. Compute M^2 and $\det(M)$. Is this consistent with your previous answers?

Problem 4 ($\sim$ 7 points)**.**

Let $\mathbb{C}$ denote the set of complex numbers and $\mathcal{M} = \mathcal{M}_2(\mathbb{R})$ denote the set of 2×2 matrices with real coefficients. Consider the map

$$f : \mathbb{C} \to \mathcal{M}$$
$$z = a + ib \mapsto \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

(1) Show that f is a homomorphism from $(\mathbb{C}, +)$ to $(\mathcal{M}, +)$.

(2) Show that f is a homomorphism from $(\mathbb{C}, \times)$ to $(\mathcal{M}, \times)$.

(3) Show that f is injective.

(4) Show that f is not surjective.

(5) Let $C \subseteq \mathcal{M}$ denote the image of f, in other words $C = \{f(z), z \in \mathbb{C}\}$. Show that the map

$$\begin{aligned} \tilde{f} \colon \mathbb{C} &\to C \\ z &\mapsto f(z) \end{aligned}$$

is an isomorphism from $(\mathbb{C}, +)$ to $(\mathcal{M}, +)$ and from $(\mathbb{C}, \times)$ to $(\mathcal{M}, \times)$.