RUTGERS UNIVERSITY | NEWARK

21:640:403
Complex variables

Fall 2016

Test #1

Monday, October 10 2016

NAME: ______________________

Please write clearly and properly. Justify all your answers.

Problem	Grade
1	
2	
3	
Total	

Problem 1 (~4 points)**.** Solve the equation:

$$z^2 + z + \frac{1 - 2i}{4} = 0 \; .$$

Problem 2 (~7 points)**.** Consider the complex exponential function:

$$\begin{aligned} \exp\colon \mathbb{C} &\to \mathbb{C} \\ z &\mapsto e^z \end{aligned}$$

(1) What is the domain of definition of the function exp? What is its target?

(2) Consider the complex numbers $z_1 = 0$, $z_2 = i\pi$, $z_3 = 5i\pi$ and $z_4 = 3 - i\pi/2$. Compute the image of each of these complex numbers by the function exp. Write your answers in algebraic form.

(3) Consider the complex numbers $y_1 = 1$, $y_2 = 0$, $y_3 = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$ and $y_4 = 2i$. Find all the preimages of each of these complex numbers by the function exp. Write your answers in algebraic form.

(4) Is the function exp injective? Is it surjective? Is it bijective? Justify all your answers.

Problem 3 (~13 points)**.**

(1) (i) Let x be a real number. Define the set $V_x \subseteq \mathbb{C}$ as follows:

$$V_x = \{z \in \mathbb{C}\colon\ Re(z) = x\} \ .$$

Sketch V_x in the complex plane for $x = 2$.

(ii) Let y be a real number. Define the set $H_y \subseteq \mathbb{C}$ as follows:

$$H_y = \{z \in \mathbb{C}\colon\ Im(z) = y\} \ .$$

Sketch H_y in the complex plane for $y = 1$.

(iii) Let x_1 and x_2 be two real numbers. Define the set $V_{x_1,x_2} \subseteq \mathbb{C}$ as follows:

$$V_{x_1,x_2} = \{z \in \mathbb{C} \colon x_1 \leqslant Re(z) \leqslant x_2\} \ .$$

Sketch V_{x_1,x_2} in the complex plane for $(x_1, x_2) = (-1, 2)$.

(iv) Let y_1 and y_2 be two real numbers. Define the set $H_{y_1,y_2} \subseteq \mathbb{C}$ as follows:

$$H_{y_1,y_2} = \{z \in \mathbb{C} \colon y_1 \leqslant Im(z) \leqslant y_2\} \ .$$

Sketch H_{y_1,y_2} in the complex plane for $(y_1, y_2) = (1, 3)$.

(v) Let x_1, x_2, y_1, y_2 be four real numbers. Define the set $R_{x_1,x_2,y_1,y_2} \subseteq \mathbb{C}$ as follows:

$$R_{x_1,x_2,y_1,y_2} = V_{x_1,x_2} \cap H_{y_1,y_2} \ .$$

Sketch R_{x_1,x_2,y_1,y_2} in the complex plane for $(x_1, x_2, y_1, y_2) = (-1, 2, 1, 3)$.

(2) (i) Is V_{x_1,x_2} open? Is it closed? Is it compact? Is it connected?

(ii) Is R_{x_1,x_2,y_1,y_2} convex? Is it star-shaped? Is it simply connected? Is it path-connected?

(3) (i) What is the image of V_x by the exponential function? Sketch $\exp(V_x)$ in the complex plane for some value of x of your choosing.

(ii) What is the image of H_y by the exponential function? Sketch $\exp(H_y)$ in the complex plane for some value of y of your choosing.

(iii) What is the image of V_{x_1,x_2} by the exponential function? Sketch $\exp(V_{x_1,x_2})$ in the complex plane for some values of x_1 and x_2 of your choosing.

(iv) What is the image of H_{y_1,y_2} by the exponential function? Sketch $\exp(H_{y_1,y_2})$ in the complex plane for some values of y_1 and y_2 of your choosing.

(v) What is the image of R_{x_1,x_2,y_1,y_2} by the exponential function? Sketch $\exp(R_{x_1,x_2,y_1,y_2})$ in the complex plane for some values of x_1, x_2, y_1 and y_2 of your choosing.

(4) (i) Is $\exp(R_{x_1,x_2,y_1,y_2})$ open? Is it closed? Is it compact? Is it connected?

(ii) Is $\exp(R_{x_1,x_2,y_1,y_2})$ convex? Is it star-shaped? Is it simply connected? Is it path-connected?