Homework exercises #6

Problem 1. For each of the functions f of a complex variable z defined below, answer the following questions:

(1) What is the domain of definition of f?

(2) Without much explanations, find the derivatives $\frac{\partial f}{\partial z}$ and $\frac{\partial f}{\partial \bar{z}}$ ("take a guess").

(3) Express f as a function of two real variables (x, y), where $z = x + iy$. Compute the derivatives $\frac{\partial f}{\partial z}$ and $\frac{\partial f}{\partial \bar{z}}$ rigorously (using their definition in terms of $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$). Check that what you found is consistent with the previous question.

(4) Is f holomorphic? If yes, what is $f'(z)$?

(a)
$$f(z) = 1 + z + z^2$$

(b)
$$f(z) = 2\,\mathrm{Im}(z)$$

(c)
$$f(z) = |z|^4$$

(d)
$$f(z) = \frac{1}{z - i}$$

(e)
$$f(z) = e^{\bar{z}}$$

Problem 2. Consider the following function:

$$\begin{aligned} f\colon \mathbb{R}^2 \approx \mathbb{C} &\to \mathbb{C} \\ (x, y) &\mapsto \sin x \cosh y + i \cos x \sinh y\,. \end{aligned}$$

We recall that cosh *and* sinh *are the functions defined by:*

$$\begin{aligned} \cosh : \mathbb{R} &\to \mathbb{R} \\ x &\mapsto \frac{e^x + e^{-x}}{2} \end{aligned} \qquad \begin{aligned} \sinh : \mathbb{R} &\to \mathbb{R} \\ x &\mapsto \frac{e^x - e^{-x}}{2} \end{aligned}$$

(1) What is the domain of definition of f?

(2) Compute the derivatives $\dfrac{\partial f}{\partial z}$ and $\dfrac{\partial f}{\partial \overline{z}}$.

(3) Is f holomorphic? What is $f'(z)$?

(4) Check that $f(z) = \sin(z)$ and $f'(z) = \cos(z)$.

We recall that the complex cos *and* sin *are the functions defined by:*

$$\begin{aligned} \cos : \mathbb{C} &\to \mathbb{C} \\ z &\mapsto \frac{e^{iz} + e^{-iz}}{2} \end{aligned} \qquad\qquad \begin{aligned} \sin : \mathbb{C} &\to \mathbb{C} \\ z &\mapsto \frac{e^{iz} - e^{-iz}}{2i} \end{aligned}$$

Problem 3. Let f be a *Möbius transformation*, *i.e* a function given by:

$$f(z) = \frac{az + b}{cz + d}$$

where a, b, c, d be four complex numbers.

Möbius transformations were already discussed to some extent in Problem 3 and Problem 4 of Homework #5.

(1) What is the domain of definition of f?

(2) Show that if $ad - bc = 0$, then f is constant. *Hint: write* $f(z) = \dfrac{a}{c} + g(z)$.

(3) Show that if $ad - bc \neq 0$, then $f : D \to f(D)$ is bijective, determine $f(D)$, and show that the inverse of f is the function

$$\begin{aligned} g\colon f(D) &\to D \\ z &\mapsto \frac{dz - b}{-cz + a}\ . \end{aligned}$$

By definition, the inverse of f is the unique function $g : f(D) \to D$ such that $f(z) = w$ if and only if $z = g(w)$.

(4) Is f holomorphic? Is f entire?

(5) It is a general fact that the inverse of a bijective holomorphic function is also holomorphic. Check that this applies in the situation of this problem.

(6) Compute $f'(z)$.

(7) It is a general fact that if $f : U \to \mathbb{C}$ is holomorphic, where $U \subseteq \mathbb{C}$ is open and connected, then f is a constant function if and only if f' vanishes everywhere. Check that this applies in the situation of this problem.

Problem 4. A function f defined on some open set $U \subseteq \mathbb{R}^2 \approx \mathbb{C}$ with real or complex values is called *harmonic* when $\Delta f = 0$ everywhere in U, where the *Laplacian* Δ is the differential operator given by:

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

Remark: Of course, f first has to be a function such that the second partial derivatives $\frac{\partial^2 f}{\partial x^2}$ and $\frac{\partial^2 f}{\partial y^2}$ exist in order to be harmonic, but we will not be concerned about this issue in this problem (we will only consider twice differentiable functions).

(1) Are the following functions harmonic?

$$u_1(x, y) = 3$$
$$u_2(x, y) = x^2 + y^2$$
$$u_3(x, y) = 1 + xy + \cos(x)\sinh(y)$$

(2) Are the following functions harmonic?

$$f_1(z) = z^4$$
$$f_2(z) = |z|^4$$
$$f_3(z) = e^z$$

(3) Show that

$$\Delta = 4\,\frac{\partial^2}{\partial z\,\partial \overline{z}} = 4\,\frac{\partial^2}{\partial \overline{z}\,\partial z}\ .$$

In other words, check that for any twice-differentiable function $f : U \to \mathbb{C}$:

$$\Delta(f) = 4\,\frac{\partial}{\partial z}\left(\frac{\partial f}{\partial \overline{z}}\right) = 4\,\frac{\partial}{\partial \overline{z}}\left(\frac{\partial f}{\partial z}\right)\ .$$

(4) Show that if f is a holomorphic function or a anti-holomorphic function, then f is harmonic.

(5) Is the following function harmonic?

$$z \mapsto \frac{e^{2\overline{z}} + \overline{z}^3}{2i - \overline{z}^2} + 8z\overline{z} + \frac{6\sin(z)}{1 + \cos(z)^2}\ .$$

Additional exercises

If you have extra time, now or when you come back to this sheet of exercises in the future, you may look at the following exercises from the textbook:

Exercise 2.17

Exercise 2.20

Exercise 2.24

Exercise 2.26