Homework exercises #5

Problem 1.
Study the convergence of the sequence of complex numbers $w_n = e^{2i\pi/n}$ in two different ways:

(1) Use a direct argument.

(2) Use the continuity of the exponential function.

Problem 2.
Consider the function f of a complex variable z defined by:

$$f(z) = \frac{z^4 + 2e^{6z^2-1}}{(z-4)^3\,(z-2i)^7}\ .$$

Consider the set $K \subseteq \mathbb{C}$ defined by:

$$K = \{z \in \mathbb{C} : \ |Re(z)| \leqslant 2,\ |Im(z)| \leqslant 1\}\ .$$

Show that the set $f(K) \subseteq \mathbb{C}$ is compact and connected.

Problem 3.
Let a, b, c, d be four complex numbers. Consider the function f of a complex variable z defined by:

$$f(z) = \frac{az+b}{cz+d}\ .$$

This type of function plays a major role in complex analysis and geometry. It is called a *Möbius transformation* (also *homography* or *projective linear map*).

(1) What is the domain of definition of f? Let us denote it by D in what follows. *Answer:* $D = \mathbb{C} \setminus \{-\frac{d}{c}\}$.

(2) What is the target of f? What is the image of f? *By definition, the image of f is the set $f(D)$. Answer:* $f(D) = \mathbb{C} \setminus \{\frac{a}{c}\}$.

(3) Show that if $ad - bc = 0$, then f is constant.

(4) Show that if $ad - bc \neq 0$, then:
   (i) $f : D \to f(D)$ is bijective.
   (ii) The inverse of f is the function

$$g\colon f(D) \to D$$
$$z \mapsto \frac{dz - b}{-cz + a}\ .$$

By definition, the inverse of f is the unique function $g : f(D) \to D$ such that $f(z) = w$ if and only if $z = g(w)$.

Problem 4.
Let f be the function as in problem 3. We assume in this problem that $ad - bc \neq 0$. Consider the functions f_1, f_2, f_3 and f_4 defined by:

$$f_1(z) = z + \frac{d}{c}$$
$$f_2(z) = \frac{1}{z}$$
$$f_3(z) = \frac{bc - ad}{c^2}\, z$$
$$f_4(z) = z + \frac{a}{c}$$

(1) Show that $f = f_4 \circ f_3 \circ f_2 \circ f_1$.

(2) Show that f_1, f_2, f_3 and f_4 are all continuous. Conclude that f is continuous. Was there a more simple argument to show that f is continuous?

(3) Show that the image of a straight line by an affine function is always a straight line, and the image of a round circle by an affine function is always a round circle.

(4) Show that the image of a straight line by the function $z \mapsto \frac{1}{z}$ is either a straight line or a round circle. Also show that the image of a round circle by the function $z \mapsto \frac{1}{z}$ is either a straight line or a round circle. *You may skip this question and accept its result.*

(5) Conclude that the image of a straight line or a round circle by f is always a straight line or a round circle.

(6) What can you predict about the topological properties of the image of a round circle by f?

Additional exercises from the textbook
If you have extra time, now or when you come back to this sheet of exercises in the future, you may look at the following exercises from the textbook:
Exercise 2.3
Exercise 2.7
Exercise 2.8