RUTGERS UNIVERSITY | NEWARK

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Complex variables

Fall 2016

Homework exercises #4

Problem 1.

What is the image of a rectangle in the complex plane by the exponential function?

More precisely, consider the complex exponential function $\exp\colon \mathbb{C} \to \mathbb{C}$, and let $R \subset \mathbb{C}$ be the compact rectangle given by:

$$R = \{z \in \mathbb{C} \colon a \leqslant x \leqslant b, c \leqslant y \leqslant d\}$$

where a, b, c, d are real constants.
Describe the set $\exp(R)$ in algebraic and/or polar coordinates, and sketch it in the complex plane.

Problem 2. Find a domain U as big as possible, so that the restriction of the exponential function

$$\exp : U \to \mathbb{C}$$

is an injective function.

Problem 3.

(1) Prove that affine function $\mathbb{C} \to \mathbb{C}$ is bijective. *Recall than an affine function is a degree* 1 *polynomial function.*
(2) Prove that the conjugation function $\mathbb{C} \to \mathbb{C}$, $z \mapsto \overline{z}$ is bijective.

Problem 4.
Let n be a positive integer and consider the function

$$\begin{aligned} f\colon \mathbb{C} &\to \mathbb{C} \\ z &\mapsto z^n \end{aligned}$$

(1) What is the domain of definition of f? What is its target?

(2) Determine $f(A)$, where:
   (i) $A = \mathbb{R}$ is the real line.
   (ii) $A = e^{i\theta}\mathbb{R}^+$ is the half-line through the origin with angle θ.
   (iii) $A = C(0, r)$ is a circle centered at the origin.
   (iv) $A = \{z \in D(0,1)\colon 0 \leqslant Arg(z) \leqslant 2\pi/n\}$ is an angular sector in the unit disk with opening $2\pi/n$.

(3) (i) Determine $f^{-1}(\{0\})$ (in other words, find all the preimages of 0).
   (ii) Determine $f^{-1}(\{1\})$.
   (iii) Let $w \in \mathbb{C}^*$ be a nonzero complex number. Determine $f^{-1}(\{w\})$. *Hint: work with polar forms.*

(4) Is f injective? Is it surjective? Is it bijective?

Problem 5.
Consider an affine function

$$\begin{aligned} f\colon \mathbb{C} &\to \mathbb{C} \\ z &\mapsto az + b \end{aligned}$$

where a and b are complex constants, with $a \neq 0$.

(1) Prove that f is bijective.

(2) Let $a = \rho e^{i\theta}$ denote the polar form of a. Consider the following functions:

$$f_1\colon \begin{array}{ll} \mathbb{C} & \to \mathbb{C} \\ z & \mapsto \rho z \end{array} \qquad f_1\colon \begin{array}{ll} \mathbb{C} & \to \mathbb{C} \\ z & \mapsto e^{i\theta} z \end{array} \qquad f_3\colon \begin{array}{ll} \mathbb{C} & \to \mathbb{C} \\ z & \mapsto z + b \end{array}$$

Show that $f = f_3 \circ f_2 \circ f_1$.

(3) What is the image of a square by f? *Hint: study the functions f_1, f_2 and f_3 independently.*