RUTGERS
UNIVERSITY | NEWARK

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Complex variables

Fall 2016

Homework exercises #3

Problem 1 (Connectedness)**.** For each one of the sets $C_k \subseteq \mathbb{C}$ $(1 \leqslant k \leqslant 12)$ defined below, answer the following questions:

- Draw a sketch of the set C_k in the complex plane.
- Is C_k convex?
- Is C_k star-shaped?
- Is C_k simply connected?
- Is C_k path-connected?
- Is C_k connected?

Briefly explain your answers.

(1) $C_1 = \mathbb{C}$

(2) $C_2 = D(0,1)$

(3) $C_3 = \overline{D(0,1)}$

(4) $C_4 = C(0,1) = C_3 - C_2$

(5) $C_5 = D^*(0,1) = D(0,1) - \{0\}$

(6) $C_6 = C(0,1) - \{1\}$

(7) $C_7 = C(0,1) - \{-1,1\}$

(8) $C_8 = \{z \in \mathbb{C}\colon\ Re(z) < -1\}$

(9) $C_9 = D(0,2) - D(1,1/2)$

(10) $C_{10} = D(0,2) - D(1,1)$

(11) $C_{11} = D(0,2) - (D(1,1) \cup D(-1,1))$

(12) $C_{12} = \mathbb{C} - \mathbb{R}^-$

Problem 2 (Homotopic paths)**.** Let $n \in \mathbb{Z}$ be an integer. Denote by γ_n the path in $\mathbb{C}$ defined by:

$$\begin{aligned} \gamma_n \colon [0,1] &\to \mathbb{C} \\ t &\mapsto e^{2n\pi i t} \end{aligned}$$

(1) Draw the paths $\gamma_1, \gamma_{-1}, \gamma_0, \gamma_2, \gamma_3$.

(2) For which values of n is γ_n a loop?

(3) For which values of n and m are the paths γ_n and γ_m homotopic in $D(0,2)^* = D(0,2) - \{0\}$?

(4) For which values of n and m are the paths γ_n and γ_m homotopic in $D(0,2)$?

Problem 3 (Solving a quadratic equation)**.** Consider the following equation in z:

$$2\,z^2 + \sqrt{6}\,z + \frac{1 - 2i\sqrt{3}}{4} = 0$$

Solve this equation. In order to do so, you may use the following guide:

(1) What kind of equation is this? How many solutions do you expect to find?

(2) Compute the discriminant Δ of this equation.

(3) Find the square roots of Δ, *i.e.* the two complex numbers δ and $-\delta$ that square to Δ. In order to find them, you may want to work with polar forms. Check that $\delta^2 = \Delta$ using algebraic forms.

(4) Find the two solutions α_1 and α_2.

(5) Check that $2(z - \alpha_1)(z - \alpha_2) = 2\,z^2 + \sqrt{6}\,z + \frac{1-2i\sqrt{3}}{4}$.

Problem 4 (A special quadratic equation)**.** Solve the following quadratic equation:

$$z^2 + z + 1 = 0$$

Show that the 3rd-root of unity $j = e^{2i\pi/3}$ is one of the roots.

Problem 5 (Polynomial with prescribed roots)**.** Find a polynomial whose roots are 0 with multiplicity 2, $1 - i$ with multiplicity 1 and $1 + i$ with multiplicity 1. Is such a polynomial unique?

Problem 6 (A special algebraic equation of degree n)**.** Let $n \in \mathbb{N}$ be a positive integer. Consider the following equation:

$$z^n = 1$$

(1) How many solutions are there? What are they called?

(2) Let $z = e^{2i\pi/n}$. Show that the solutions are $1, z, z^2, \ldots, z^{n-1}$.

(3) Recall that for any complex number $z \neq 1$, the following identity holds:

$$1 + z + z^2 + \cdots + z^{n-1} = \frac{1 - z^n}{1 - z} .$$

Show that the sum of all the n-th roots of unity is equal to zero. Is this consistent with problem 4?