RUTGERS
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Complex variables

Fall 2016

Homework exercises set #2

Problem 1. For each of the following sequences of complex numbers $(z_n)_{n\in\mathbb{N}}$, determine whether the sequence is converging, and when it is, find its limit. Prove your answers.

(1) $z_n = e^{i/n}$

(2) $z_n = e^{ni\theta}$, where θ is some real number.

(3) $z_n = \frac{1}{n} e^{ni\theta}$, where θ is some real number.

(4) $z_n = \frac{1 + ni}{n}$.

(5) $z_n = {z_0}^n$, where z_0 is some complex number.

(6) $z_n = n(1 - e^{i\theta/n})$, where θ is some real number.

Problem 2. For each one of the sets $A_k \subseteq \mathbb{C}$ $(1 \leqslant k \leqslant 12)$ defined below, answer the following questions:

- Draw a sketch of the set A_k in the complex plane.
- Is A_k open?
- Is A_k closed?
- Is A_k compact?
- Is A_k connected?
- Is A_k simply connected? [Note: Ignore this question for now.]

Briefly explain your answers.

(1) $A_1 = \mathbb{C}$

(2) $A_1 = \left\{z \in \mathbb{C}^* : \frac{\pi}{6} < Arg(z) < \frac{\pi}{3}\right\}.$

(3) $A_2 = \{z \in \mathbb{C}\colon\ Im(z) > 0\}$

(4) $A_3 = \{z \in \mathbb{C}\colon\ Im(z) \geqslant 0\}$

(5) $A_4 = D(i, 1)$

(6) $A_5 = \overline{D(-1+i, 1)}$

(7) $A_6 = \{z \in \mathbb{C}\colon\ 1 \leqslant Re(z) \leqslant 3,\ 0 \leqslant Im(z) \leqslant 2\}$

(8) $A_7 = \{z \in \mathbb{C}\colon\ -1 \leqslant Re(z) \leqslant 2,\ Im(z) = 1\}$

(9) $A_8 = A_5 \cup A_6$

(10) $A_9 = A_4 \cup A_6$

(11) $A_{10} = A_5 \cup A_6 \cup A_7$

(12) $A_{10} = D(0, 5) - A_8$

Problem 3. Let z_0 and z_1 be any two complex numbers. Define a map

$$\begin{aligned} \gamma\colon [0,1] &\to \mathbb{C} \\ t &\mapsto (1-t)z_0 + t\, z_1\ . \end{aligned}$$

(1) Show that γ is a continuous path from z_0 to z_1.

(2) Show that the image of γ is the line segment $[z_0, z_1]$.

(3) A set $C \subseteq \mathbb{C}$ is called *convex* if it has the property that for any two points $z_0 \in \mathbb{C}$ and $z_1 \in \mathbb{C}$, the line segment $[z_0, z_1]$ is a subset of C. Prove that any convex set is connected.

(4) Draw an example of a set which is connected but not convex.

(5) A set $C \subseteq \mathbb{C}$ is called *star-shaped* if it has the property that there exists a point $z_0 \in \mathbb{C}$ such that for any point $z_1 \in \mathbb{C}$, the line segment $[z_0, z_1]$ is a subset of C. Prove that any star-shaped set is connected.

(6) Draw an example of a set which is star-shaped but not convex.