Homework exercises #12

Problem 1.
Let $f : U \to \mathbb{C}$ be a holomorphic function. Let $D(z_0, r) \subseteq U$. Assume that $m \leqslant |f(z)| \leqslant M$ for every $z \in C(z_0, r)$ (where m and M are some nonnegative real constants). Show that $m \leqslant |f(z_0)| \leqslant M$.

Problem 2. In this exercise, we let $C(z_0, r)$ denote any positively oriented parametrization of the circle with center z_0 and radius r in the complex plane. Using Cauchy's integral formula, compute the following line integrals. Carefully justify your computation.

(1)
$$I_1 = \int_{C(0,1)} \frac{\cos z \, dz}{z}$$

(2)
$$I_2 = \int_{C(0,1)} \frac{\sin z \, dz}{z}$$

(3)
$$I_3 = \int_{C(0,2)} \frac{dz}{z^2 + 1}$$

(4)
$$I_4 = \int_{C(0,2)} \frac{e^{iz} \, dz}{z^2 + 1}$$

(5)
$$I_5 = \int_{C(0,2)} \frac{\cos z \, dz}{z^3 + 9z}$$

Here are the answers you are supposed to find: (1) $I_1 = 2i\pi$ *(2)* $I_2 = 0$ *(3)* $I_3 = 0$ *(4)* $I_4 = -2\pi \sinh(1)$ *(5)* $I_5 = \frac{2i\pi}{9}$.

Additional exercises
Here are additional exercises from the textbook:
Exercise 4.34
Exercise 4.35
Exercise 4.36

Finally, here is an optional additional exercise:

Problem 3. Show the following theorem, called the *maximum principle*:

Theorem. Let $f : U \to \mathbb{C}$ be a holomorphic function, where U is an open connected set in the complex plane. Show that if the modulus of f admits a local maximum in U, then f is constant.

Hint: Start by showing that if $|f|$ admits a local maximum at $z_0 \in U$, then $|f|$ is constant on any circle $C(z_0, r)$, provided $D(z_0, r) \subseteq U$.