Quiz #4: Solutions

Monday, October 9 2017

Problem 1.

Let us write a proof of the following theorem:

Theorem. *For any integer $n \in \mathbb{Z}$, n is even if and only if n^2 is even.*

In order to prove this (universally quantified) biconditional proposition, we first prove one implication, then we prove the converse implication.

Step 1: Let us prove the first implication: $\forall n \in \mathbb{Z} \quad (n \text{ even } \rightarrow n^2 \text{ even})$.

We produce a *direct proof* of this implication. Let $n \in \mathbb{Z}$. Let us assume that n is even. The goal is to show that n^2 is even. By assumption, n is even so there exists $k \in \mathbb{Z}$ such that $n = 2k$. It follows that $n^2 = 4k^2$. Thus $n^2 = 2K$ where $K = 2k^2 \in \mathbb{Z}$. This shows that n^2 is even.

Step 1: Let us prove the converse implication: $\forall n \in \mathbb{Z} \quad (n^2 \text{ even } \rightarrow n \text{ even})$.

The first try to prove any implication (also called conditional statement) should be a direct proof, however a direct proof does not work here. Hence we try a *proof by contrapositive*. Let $n \in \mathbb{Z}$. Let us assume that n is not even. The goal is to show that n^2 is not even. By assumption, n is odd so there exists $k \in \mathbb{Z}$ such that $n = 2k + 1$. It follows that $n^2 = 4k^2 + 4k + 1$. Thus $n^2 = 2K + 1$ where $K = 2k^2 + 2k \in \mathbb{Z}$. This shows that n^2 is odd, in other words it is not even.

Problem 2.

Let us write a proof of the following theorem:

Theorem.

$$\forall a \in \mathbb{R}\, \forall b \in \mathbb{R} \quad (a \in \mathbb{Q} \ \wedge \ b \in \mathbb{Q}) \ \rightarrow \ a + b \in \mathbb{Q}\,.$$

Let us write a direct proof. Let us assume that a and b are two real numbers that are both rational. We want to show that $a + b$ is rational. By definition, since a is rational, there exists integers p and q (with $q \neq 0$) such that $a = p/q$. Similarly, since b is rational, there exists integers p' and q' (with $q' \neq 0$) such that $b = p'/q'$. Thus we can write:

$$
\begin{aligned}
a + b &= \frac{p}{q} + \frac{p'}{q'} \\
a + b &= \frac{pq' + qp'}{qq'} \\
a + b &= \frac{p''}{q''}
\end{aligned}
$$

where $p'' = pq' + qp'$ and $q'' = qq'$. Hence we have showed that $a + b$ is the ratio of two integers, which proves that $a + b$ is rational.

The converse to this theorem is false. For instance, $a = 2 - \sqrt{2}$ and $b = \sqrt{2}$ are both irrational, even though $a + b = 1$ is rational.

Problem 3 ($\sim$ 6 points.)**.**

Let us write a proof of the following theorem:

Theorem. *There exists no smallest positive rational number.*

As suggested, we are going to write a proof by contradiction. Assume that there exists a smallest positive rational number. Call it q. Consider the number $q/2$. It is clear that $q/2$ is also rational and positive. Moreover, $q/2$ is smaller than q. This is a contradiction, because q is the smallest positive rational number by assumption.