RUTGERS
UNIVERSITY | NEWARK

Math 237
Discrete Structures

Fall 2017

Exam #2 Solutions

Problem 1.

(1) The function f_1 is not injective because, for instance, $f_1(-1) = f_1(1)$. It is not surjective either, because, for instance, -1 is an element of the codomain that has no preimage. Being neither injective nor surjective, f_1 is not bijective.

(2) The function f_2 is injective because $f(x_1) = f(x_2)$ implies that $x_1 = x_2$ for any x_1 and x_2 in $\mathbb{Z}$. Indeed, if $f(x_1) = f(x_2)$, then $-4x_1 + 3 = -4x_2 + 3$, it easily follows that $x_1 = x_2$. The function f_2 is not sujective: for instance, 0 is the codomain of f and has no preimage. The function f_2 is not bijective because it is not surjective.

Problem 2.

(1) The composition $f \circ g$ is well-defined because the codomain of g is equal to the domain of f. The function $f \circ g$ is:

$$\begin{aligned} f \circ g \colon \mathbb{R}_{\geqslant 0} &\longrightarrow \mathbb{R}_{\geqslant 0} \\ x &\longmapsto (\sqrt{x})^2 = x \end{aligned}$$

(2) The composition $g \circ f$ is well-defined because the codomain of f is equal to the domain of g. The function $f \circ g$ is:

$$\begin{aligned} g \circ f \colon \mathbb{R} &\longrightarrow \mathbb{R} \\ x &\longmapsto \sqrt{x^2} = |x| \end{aligned}$$

Problem 3.

Is the following sequence: increasing? decreasing? nonincreasing? nondecreasing? *Answer all four questions! No explanations required.*

(1) The sequence $(u_n)_{n\in\mathbb{N}}$ is not increasing, not decreasing, not nonincreasing, nondecreasing.

(2) The sequence $(u_n)_{n\in\mathbb{N}}$ is not increasing, not decreasing, not nonincreasing, not nondecreasing.

(3) The sequence $(w_n)_{n\in\mathbb{N}}$ is not increasing, not decreasing, not nonincreasing, not nondecreasing.

Problem 4.

λ (empty string), 0, 1, 00, 01, 10, 11, 000, 001, 010, 011, 100, 101, 110, 111.

Problem 5.

(1) We need to prove that $\mathcal{R}$ is reflexive, symmetric and transitive:
- $\mathcal{R}$ is reflexive because it is true that for any $x \in X$, $x = x \pmod{n}$. Indeed, n divides 0.
- $\mathcal{R}$ is symmetric because it is true that for any $x \in X$ and $y \in X$, $x = y \pmod{n}$ if and only if $y = x \pmod{n}$. Indeed, if n divides $x - y$, then n divides $y - x$, and conversely.
- $\mathcal{R}$ is transitive because it is true that for any $x \in X$, $y \in X$, and $y \in X$, if $x = y \pmod{n}$ and $y = z \pmod{n}$, then $x = z \pmod{n}$. Indeed, if n divides $x-y$ and n divides $y-z$, then n divides $x-z$ (this is because $x-z = (x-y)+(y-z)$).

(2) The equivalence class of 0 is the set of positive even integers. The equivalence class of 1 is the set of positive odd integers. The equivalence class of 2 is the set of positive even integers. The equivalence class of 3 is the set of positive odd integers.

(3) There are n equivalence classes: $[0], [1], \ldots, [n-1]$.

Problem 6.

(1) **Input:** X (sequence of integers), x (integer)
Output: *out* (integer)

Here is the algorithm written in pseudocode:

```
custom_sum(X,x)
{
    n = length(X);
    out = 0;
    for i=1 to i=n
    {
        if X(i)>x
        {
            out = out + X(i);
        }
    }
    return out;
}
```

(2) Here is the trace of the algorithm for $X = (4, 3, 10, 5)$ and $x = 5$:

$n = 4$
$out = 0$
$i = 1$
$i = 2$
$i = 3$
$out = 10$
$i = 4$
return10

(3) For this algorithm, one operation = one addition. In the worst case scenario, there are n additions, where n is the length of X. Therefore the complexity is equal to n.

Problem 7.

(1) $4n^3 - n^2 + 1 = \mathrm{O}(n^3)$ TRUE

(2) $4n^3 - n^2 + 1 = \Theta(n^3)$ TRUE

(3) $4n^3 - n^2 + 1 = \mathrm{o}(n^3)$ FALSE

(4) $4n^3 - n^2 + 1 = \mathrm{O}(n^4)$ TRUE

(5) $4n^3 - n^2 + 1 = \mathrm{o}(n^4)$ TRUE

(6) $n^{1000} + 100n^3 = o(e^n)$ TRUE

(7) $n^2 \log n = o(n^3)$ TRUE

(8) $n^2 \log n = \Theta(n^2)$ FALSE