Exam #1 Solutions

Monday, October 16 2017

Problem 1.

In order to prove this (universally quantified) biconditional proposition, we first prove one implication, then we prove the converse implication.

Step 1: Let us prove the first implication: $\forall n \in \mathbb{Z} \quad (n \text{ odd } \rightarrow n^2 \text{ odd})$.

We produce a *direct proof* of this implication. Let $n \in \mathbb{Z}$. Let us assume that n is odd. The goal is to show that n^2 is odd. By assumption, n is odd so there exists $k \in \mathbb{Z}$ such that $n = 2k + 1$. It follows that $n^2 = 4k^2 + 4k + 1$. Thus $n^2 = 2K$ where $K = 2k^2 + 2k \in \mathbb{Z}$. This shows that n^2 is odd.

Step 2: Let us prove the converse implication: $\forall n \in \mathbb{Z} \quad (n^2 \text{ odd } \rightarrow n \text{ odd})$.

The first try to prove any implication (also called conditional statement) should be a direct proof, however a direct proof does not work here. Hence we try a *proof by contrapositive*. Let $n \in \mathbb{Z}$. Let us assume that n is not odd. The goal is to show that n^2 is not odd. By assumption, n is even so there exists $k \in \mathbb{Z}$ such that $n = 2k$. It follows that $n^2 = 4k^2$. Thus $n^2 = 2K$ where $K = 2k^2 \in \mathbb{Z}$. This shows that n^2 is even, in other words it is not odd.

Problem 2.

The proposition is false, because $n = 4$ is a counter-example (among others): $4! + 1 = 25$ is not prime (it is divisible by 5).

Problem 3.

In order to prove this (universally quantified) biconditional proposition, we first prove one implication, then we prove the converse implication.

Step 1: Let us prove the first implication: $\forall A \forall B \quad (A \subseteq B \rightarrow \overline{B} \subseteq \overline{A})$.

We produce a *direct proof* of this implication. Let A and B be any sets. Let us assume that $A \subseteq B$. The goal is to show that $\overline{B} \subset \overline{A}$. To that end we show that any element of $\overline{B}$ is an element of $\overline{A}$: let $x \in \overline{B}$. By definition, $x \notin B$. Therefore $x \notin A$: indeed, if x was an element of A, it would be an element of B since $A \subseteq B$ by assumption. Therefore $x \in \overline{A}$.

Step 2: Let us prove the converse implication: $\forall A \forall B \quad (\overline{B} \subseteq \overline{A} \rightarrow A \subseteq B)$.

We produce a *direct proof* of this implication. Let A and B be any sets. Let us assume that $\overline{B} \subset \overline{A}$. The goal is to show that $A \subseteq B$. To that end we show that any element of A is an element of B: let $x \in A$. Let us argue by contradiction that $x \in B$: if x was not element of B, then x would be an element of $\overline{B}$. But $\overline{B} \subset \overline{A}$, so x would be an element of $\overline{A}$. This contradicts the assumption that $x \in A$.

Problem 4.

Let us write a proof by contrapositive of this (universally quantified) conditional proposition. Assume that it is not the case that a is irrational or b is irrational. In other words, assume that a and b are both rational. Our goal is to show that $a + b$ is rational. By definition, since a is rational, there exists integers p and q (with $q \neq 0$) such that $a = p/q$. Similarly, since b is rational, there exists integers p' and q' (with $q' \neq 0$) such that $b = p'/q'$. Thus we can write:

$$
\begin{aligned}
a + b &= \frac{p}{q} + \frac{p'}{q'} \\
a + b &= \frac{pq' + qp'}{qq'} \\
a + b &= \frac{p''}{q''}
\end{aligned}
$$

where $p'' = pq' + qp'$ and $q'' = qq'$. Hence we have showed that $a + b$ is the ratio of two integers, which proves that $a + b$ is rational.

Problem 5.

As suggested, let us write a proof by induction. First let us rewrite the theorem in order to precisely identify the proposition that we want to show by induction:

Theorem.

$$\forall n \in \mathbb{N} \quad P(n)$$

where $P(n)$ is the proposition: "$5 | 11^n - 6$".

Any proof by induction consists of two steps:

Basis step: Let us check that the proposition $P(1)$ is true.

$P(1)$ is the proposition that $5 | 11^1 - 6$, in other words $5 | 5$. This is clearly true.

Induction step: Let us prove that $\forall n \in \mathbb{N} \quad (P(n) \to P(n+1))$.

We write a *direct proof* of this (universally quantified) implication. Let $n \in \mathbb{N}$. Assume that $P(n)$ is true. The goal is to show that $P(n+1)$ is true. By assumption, we know that $5|11^n - 6$. In other words, there exists $k \in \mathbb{Z}$ such that $11^n - 6 = 5k$. Multiplying by 11 on both sides, we get $11^{n+1} - 66 = 55k$. Adding 60 on both sides, we find $11^{n+1} - 6 = 55k + 60$. we can rewrite this $11^{n+1} - 6 = 5K$ where $K = 55 + 12 \in \mathbb{Z}$. Thus we have proved that $5|11^{n+1} - 6$, as desired.