Math 235
Calculus III

Spring 2018

Quiz #8 Solutions

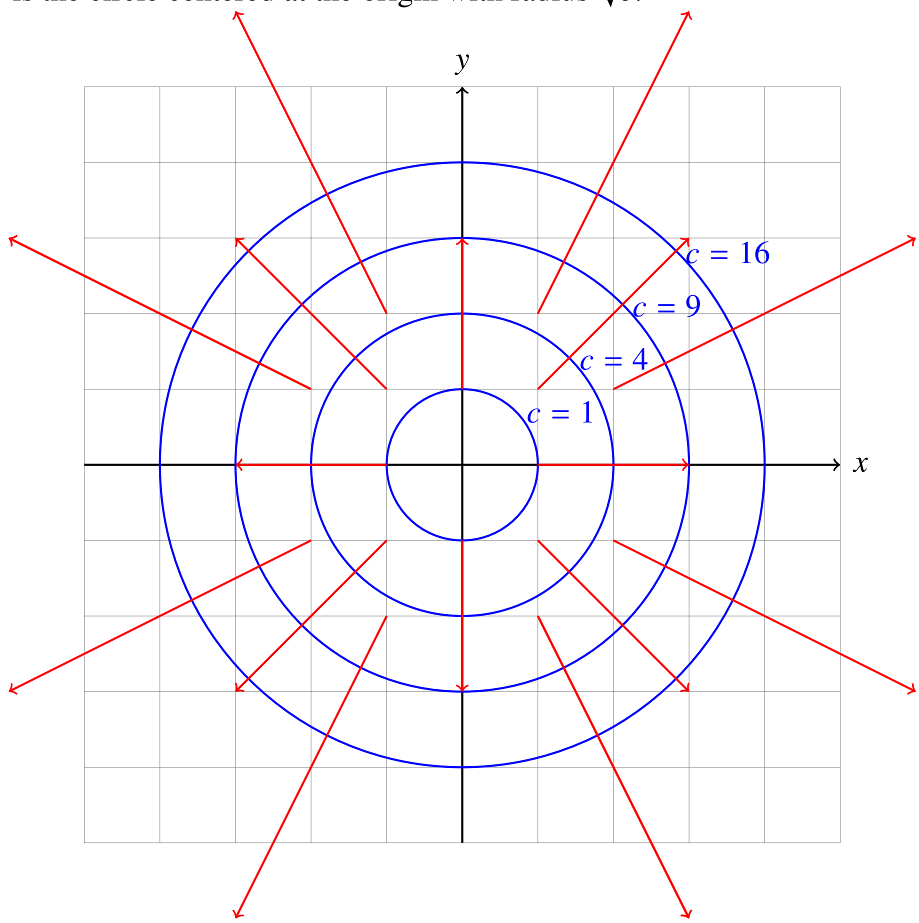
Problem 1.

(1) We compute $\vec{V}(x, y)$ as the gradient of the function φ:

$$\boxed{\vec{V}(x, y) = \vec{\nabla}\varphi(x, y) = (2x, 2y)}$$

(2) No. For instance, $\psi(x, y) = x^2 + y^2 + 1$ is another potential function for the vector field $\vec{V}(x, y)$.

(3) The equipotential curves for the vector field $\vec{V}(x, y)$ are the level curves of the function $\varphi(x, y) = x^2 + y^2$. The c-level curve of this function is the curve with equation $x^2 + y^2 = c$. If $c < 0$, the c-level curve is empty, and if $c \geqslant 0$, the c-level curve is the circle centered at the origin with radius $\sqrt{c}$.



(4) At the point $P(0,1)$, the vector $\vec{V}(P) = (2,0)$ is orthogonal to the equipotential curve through P, which is the unit circle centered at the origin in the xy-plane. This is expected because a vector field is always orthogonal to its equipotential curves.

Problem 2.

The average value of a function along a smooth curve is equal to the line integral of the function along the curve divided by the length of the curve.
The formula for the line integral of a function f along a smooth curve C parametrized by $\vec{r}(t)$ for $a \leqslant t \leqslant b$ is given by:

$$\int_C f\,ds = \int_a^b f(\vec{r}(t))\,\|\vec{r}'(t)\|\,dt$$

In the scenario of this problem, we can parametrize the line segment from A to B by the function $\vec{r}(t) = A + t\vec{AB}$, in other words $\vec{r}(t) = (1-2t, t)$ with $0 \leqslant t \leqslant 1$.
Thus we compute:

$$\begin{aligned}
\int_C f\,ds &= \int_0^1 f(\vec{r}(t))\,\|\vec{r}'(t)\|\,dt \\
&= \int_0^1 f(1-2t, t)\,\|(-2,1)\|\,dt \\
&= \int_0^1 (1-3t)\,\sqrt{5}\,dt \\
&= \sqrt{5}\left(t - \frac{3t^2}{2}\right)\Big|_{t=0}^{t=1} \\
&= -\frac{\sqrt{5}}{2}
\end{aligned}$$

It remains to divide by the length of the curve C in order to find the average value of f along C. Normally we would need to get the length of C by computing $\int_a^b \|\vec{r}'(t)\|\,dt$, but here we know that the length of C, which is the line segment from A to B, is just $L(C) = \|\vec{AB}\| = \sqrt{5}$. So the average value of f along C is:

$$\boxed{\frac{\int_C f\,ds}{L(C)} = -\frac{1}{2}}$$

Problem 3.

The formula for the circulation (*i.e.* the line integral) of a vector field $\vec{V}(x, y)$ along a curve C parametrized by $\vec{r}(t)$ for $a \leqslant t \leqslant b$ is given by:

$$\int_C \vec{V} \cdot \vec{dr} = \int_a^b \vec{V}(\vec{r}(t)) \cdot \vec{r}'(t)\, dt$$

In the scenario of this problem, we can parametrize the unit circle by the function $\vec{r}(t) = (\cos(t), \sin(t))$ for $0 \leqslant t \leqslant 2\pi$.
Thus we compute:

$$\begin{aligned}
\int_C \vec{V} \cdot \vec{dr} &= \int_0^{2\pi} \vec{V}(\vec{r}(t)) \cdot \vec{r}'(t)\, dt \\
&= \int_0^{2\pi} \vec{V}(\cos(t), \sin(t)) \cdot (-\sin(t), \cos(t))\, dt \\
&= \int_0^{2\pi} (\sin(t), \cos(t)) \cdot (-\sin(t), \cos(t))\, dt \\
&= \int_0^{2\pi} \left(\cos^2(t) - \sin^2(t)\right)\, dt \\
&= \int_0^{2\pi} \cos(2t)\, dt \\
&= \frac{\sin(2t)}{2} \Big|_{t=0}^{t=2\pi} \\
&= 0
\end{aligned}$$

Let us recap the answer:

$$\boxed{\int_C \vec{V} \cdot \vec{dr} = 0}$$

Remark: This result is predictible because $\vec{V}$ is a conservative vector field ($\vec{V} = \vec{\nabla}\varphi$ where $\varphi(x, y) = xy$), so that its circulation is zero along any closed curve.