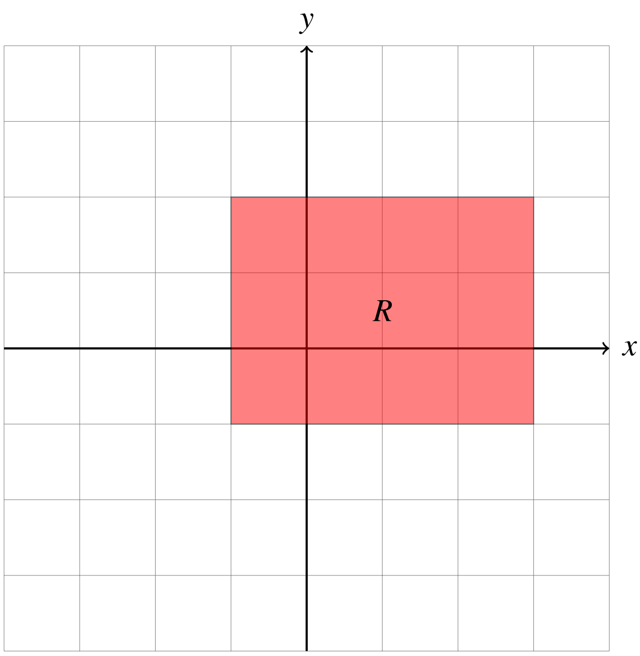
Quiz #7 Solutions

Problem 1.

(1) The domain of definition of f is $\mathbb{R}^2$, so yes it contains the rectangle R (or any other rectangle, in fact).

(2) Here is a sketch of the rectangle $R = [-1, 3] \times [-1, 2]$ in the xy-plane:



(3) We can apply Fubini's theorem:

$$\iint_R f(x, y)\,dx\,dy \;=\; \int_{y=-1}^{y=2} \left(\int_{x=-1}^{x=3} f(x, y)\,dx \right) dy \;=\; \int_{x=-1}^{x=3} \left(\int_{y=-1}^{y=2} f(x, y)\,dy \right) dx\,.$$

Following the first identity of Fubini's, theorem, we get:

$$
\begin{aligned}
\iint_R f(x,y)\,dx\,dy &= \int_{y=-1}^{y=2} \left(\int_{x=-1}^{x=3} f(x,y)\,dx \right) dy \\
&= \int_{y=-1}^{y=2} \left(\int_{x=-1}^{x=3} (1 - 2\,x\,y^2)\,dx \right) dy \\
&= \int_{y=-1}^{y=2} \left((x - x^2\,y^2) \Big|_{x=-1}^{x=3} \right) dy \\
&= \int_{y=-1}^{y=2} \left(4 - 8\,y^2 \right) dy \\
&= (4y - 8\,y^3/3) \Big|_{y=-1}^{y=2} \\
&= -12
\end{aligned}
$$

Following the second identity of Fubini's, theorem, we get:

$$
\begin{aligned}
\iint_R f(x,y)\,dx\,dy &= \int_{x=-1}^{x=3} \left(\int_{y=-1}^{y=2} f(x,y)\,dy \right) dx \\
&= \int_{x=-1}^{x=3} \left(\int_{y=-1}^{y=2} (1 - 2\,x\,y^2)\,dy \right) dx \\
&= \int_{x=-1}^{x=3} \left((y - 2\,x\,y^3/3) \Big|_{y=-1}^{y=2} \right) dx \\
&= \int_{x=-1}^{x=3} (3 - 6\,x)\;dx \\
&= (3x - 3\,x^2) \Big|_{x=-1}^{x=3} \\
&= -12
\end{aligned}
$$

As expected, we find the same value in both cases:

$$
\boxed{\iint_R f(x,y)\,dx\,dy \;=\; -12}
$$

(4) The average value of f over the rectangle R is equal to the integral of f over the rectangle R divided by the area of the rectangle R. Here the area of the rectangle is equal to $\text{Area}(R) = 4 \times 3 = 12$, so the average value of f over R is:

$$
\boxed{\text{Average}(f) = \frac{\iint_R f(x,y)\,dx\,dy}{\text{Area}(R)} = -1}
$$

Problem 2.

We can apply Fubini's theorem:

$$\iint_R f(x,y)\,dx\,dy \;=\; \int_{y=0}^{y=\pi/2} \left(\int_{x=0}^{x=\pi/2} f(x,y)\,dx \right) dy \;=\; \int_{x=0}^{x=\pi/2} \left(\int_{y=0}^{y=\pi/2} f(x,y)\,dy \right) dx\,.$$

Let us just use one of the two identities of Fubini's theorem, for instance the first one:

$$\begin{aligned}
\iint_R f(x,y)\,dx\,dy &= \int_{y=0}^{y=\pi/2} \left(\int_{x=0}^{x=\pi/2} f(x,y)\,dx \right) dy \\
&= \int_{y=0}^{y=\pi/2} \left(\int_{x=0}^{x=\pi/2} \cos(x+y)\,dx \right) dy \\
&= \int_{y=0}^{y=\pi/2} \left(\sin(x+y) \Big|_{x=0}^{x=\pi/2} \right) dy \\
&= \int_{y=0}^{y=\pi/2} \left(\sin(y+\pi/2) - \sin(y) \right) dy \\
&= \left(-\cos(y+\pi/2) + \cos(y)\right) \Big|_{y=0}^{y=\pi/2} \\
&= -\cos(\pi) + \cos(\pi/2) + \cos(\pi/2) - \cos(0) \\
&= 1 + 0 + 0 - 1 \\
&= 0
\end{aligned}$$

Let us rewrite the answer:

$$\boxed{\iint_R f(x,y)\,dx\,dy \;=\; 0}$$

Remark: We could have predicted this result without doing any computation by using a symmetry argument.