Quiz #6 Solutions

Problem 1.

(1) The function f is defined for every value of x and y, so its domain of definition is:

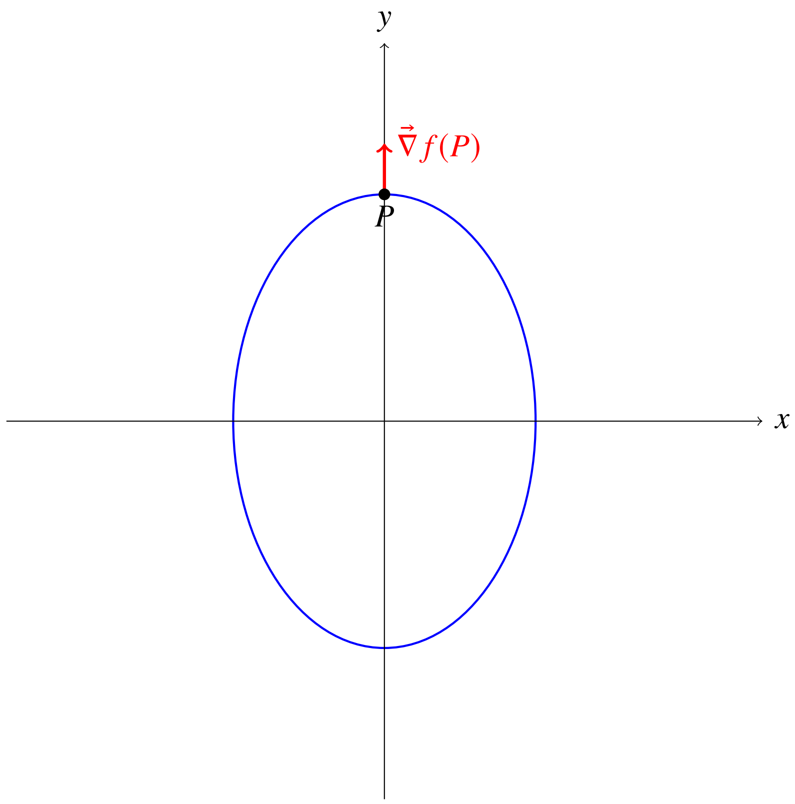
$$D = \mathbb{R}^2 .$$

(2) The graph of f is the surface with equation $z = \frac{x^2}{4} + \frac{y^2}{9}$. This is a quadric, and more precisely an elliptic paraboloid.

(3) The level curve of f through P is the curve in the xy-plane with equation $f(x, y) = c$, where $c = f(P) = 1$. Thus it is the curve with equation

$$\frac{x^2}{4} + \frac{y^2}{9} = 1 .$$

This curve is an ellipse.



(4) The partial derivatives of f are $\frac{\partial f}{\partial x}(x,y) = \frac{x}{2}$ and $\frac{\partial f}{\partial y}(x,y) = \frac{2y}{9}$, therefore the gradient is:

$$\vec{\nabla} f(x,y) = \left(\frac{x}{2}, \frac{2y}{9}\right) .$$

At $P(0,3)$, it is:

$$\vec{\nabla} f(P) = \left(0, \frac{2}{3}\right) .$$

(5) The direction of maximal rate of increase for f at P is given by $\vec{\nabla} f(P) = (0, \frac{2}{3})$. The direction of maximal rate of decrease for f at P is given $-\vec{\nabla} f(P) = (0, -\frac{2}{3})$. The direction of no change for f at P is given by any vector orthogonal to $\vec{\nabla} f(P)$, for instance $(1,0)$.
In order to have unit vectors, we can rescale these vectors by multiplying each one of them by the inverse of its magnitude. We find:

$$\begin{aligned} \vec{u} &= (0,1) \\ \vec{v} &= (0,-1) \\ \vec{w} &= (1,0) . \end{aligned}$$

(6) In order to compute these directional derivatives, we use the general formula:

$$D_{\vec{u}} f(P) = \vec{\nabla} f(P) \cdot \vec{u} .$$

In this situation we find:

$$\begin{aligned} D_{\vec{u}} f(P) &= (0, \frac{2}{3}) \cdot (0,1) = \frac{2}{3} \\ D_{\vec{v}} f(P) &= (0, \frac{2}{3}) \cdot (0,-1) = -\frac{2}{3} \\ D_{\vec{w}} f(P) &= (0, \frac{2}{3}) \cdot (1,0) = 0 . \end{aligned}$$

(7) It is the straight line with equation $y = 3$ in the xy-plane. It is the line through P directed by $\vec{w} = (1,0)$. It is indeed orthogonal to $\vec{\nabla} f(P)$, as we can check that $\vec{\nabla} f(P) \cdot \vec{w} = 0$. This is to be expected thanks to the theorem stating that at any point of the domain of definition, the gradient of the function is orthogonal to the level curve.