Quiz #4 Solutions

Problem 1.

(1) By definition, the velocity is $\vec{v}(t) = \vec{r}'(t)$. Thus we find:

$$\vec{v}(t) = \big(-\sqrt{3}\sin(t), 2\cos(t), -\sin(t)\big) .$$

By definition, the speed is $v(t) = \|\vec{v}(t)\|$. Thus we find:

$$\begin{aligned} v(t) &= \sqrt{\big(-\sqrt{3}\sin(t)\big)^2 + \big(2\cos(t)\big)^2 + \big(-\sin(t)\big)^2} \\ &= \sqrt{4\big(\cos(t)\big)^2 + 4\big(\sin(t)\big)^2} \\ &= 2 \end{aligned}$$

We can observe that this motion has constant speed.

By definition, the unit tangent vector is $\vec{T}(t) = \frac{\vec{v}(t)}{v(t)}$. Thus we find:

$$\vec{T}(t) = \left(-\frac{\sqrt{3}}{2}\sin(t), \cos(t), -\frac{1}{2}\sin(t) \right) .$$

(2) By definition, the velocity is $\vec{a}(t) = \vec{v}'(t)$. Thus we find:

$$\vec{a}(t) = \big(-\sqrt{3}\cos(t), -2\sin(t), -\cos(t)\big) .$$

(3) The Cartesian equation of a sphere is given by:

$$x^2 + y^2 + z^2 = R^2$$

where R is the radius of the sphere.
Here the coordinates $(x(t), y(t), z(t))$ of the moving point verify:

$$\begin{aligned} x(t)^2 + y(t)^2 + z(t)^2 &= \big(\sqrt{3}\cos(t)\big)^2 + \big(2\sin(t)\big)^2 + \big(\cos(t)\big)^2 \\ x(t)^2 + y(t)^2 + z(t)^2 &= 4\big(\cos(t)\big)^2 + 4\big(\sin(t)\big)^2 \\ x(t)^2 + y(t)^2 + z(t)^2 &= 4 . \end{aligned}$$

This shows that the path lies on the sphere centered at the origin with radius $R = 2$.

(4) In order to show that the path lies on the plane with Cartesian equation $x - \sqrt{3}z = 0$, we need to check that the coordinates $(x(t), y(t), z(t))$ of the moving point verify this equation. It is straightforward:

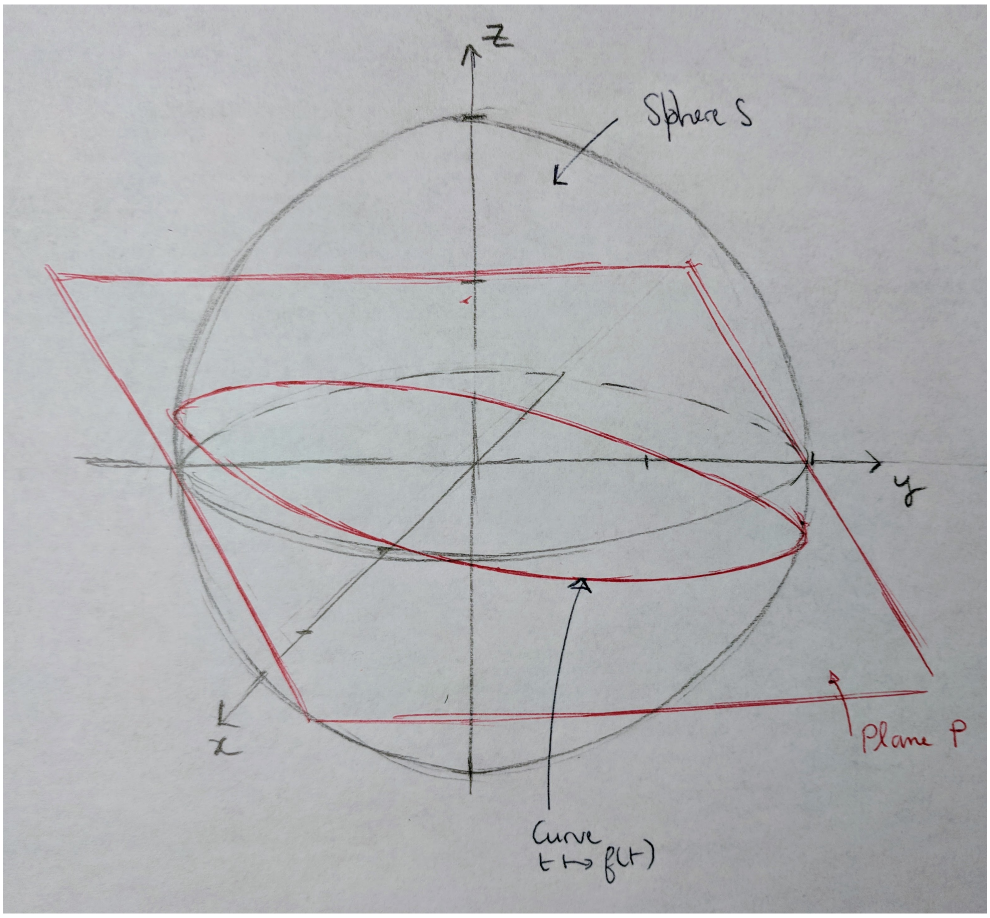
$$\begin{aligned} x(t) - \sqrt{3}z(t) &= \left(\sqrt{3}\cos(t)\right) - \sqrt{3}\left(\cos(t)\right) \\ x(t) - \sqrt{3}z(t) &= 0\,. \end{aligned}$$

This shows that the path lies on the plane with Cartesian equation $x - \sqrt{3}z = 0$.

(5) The previous two questions show that the curve is the intersection of the sphere S centered at the origin with radius $R = 2$ and the plane P with equation $x - \sqrt{3}z = 0$. In general, the intersection of a plane and a sphere is a circle (when it is not empty). Here note that the plane P goes through the origin and the sphere S is centered at the origin, thus their intersection is a circle of radius 2 centered at the origin.

In conclusion, the curve is a (slanted) circle in 3-dimensional space, centered at the origin and with radius 2.

Here is a quick sketch:



Problem 2.

We first recover the velocity $\vec{v}(t)$ by integrating the acceleration:

$$\begin{aligned}\vec{v}(t) &= \int_0^t \vec{a}(\tau)\,d\tau + \vec{v}(0)\\ &= (3t^2, 0, -2t) + (-1, 2, 0)\\ &= (3t^2 - 1, 2, -2t)\end{aligned}$$

We then recover the position vector $\vec{r}(t)$ by integrating the velocity:

$$\begin{aligned}\vec{r}(t) &= \int_0^t \vec{v}(\tau)\,d\tau + \vec{r}(0)\\ &= (t^3 - t, 2t, -t^2) + (0, 0, 1)\\ &= (t^3 - t, 2t, 1 - t^2)\end{aligned}$$

At $t = 1$, the moving point $M(t)$ has the coordinates: $(0, 1, 0)$.