Exam #2 Solutions

Problem 1.

(1) f is defined for all x and y, so $D = \mathbb{R}^2$.

(2) The graph of f is the surface with equation $z = x^2 - y^2$. It is a quadric, more precisely a hyperbolic paraboloid.

(3) The level curve through the origin is the curve with equation $f(x, y) = f(0, 0)$ in the xy-plane, *i.e.* $x^2 - y^2 = 0$. Since $x^2 - y^2 = (x - y)(x + y)$, it is the union of the two straight lines $x - y = 0$ and $x + y = 0$ (see Figure 1 below).

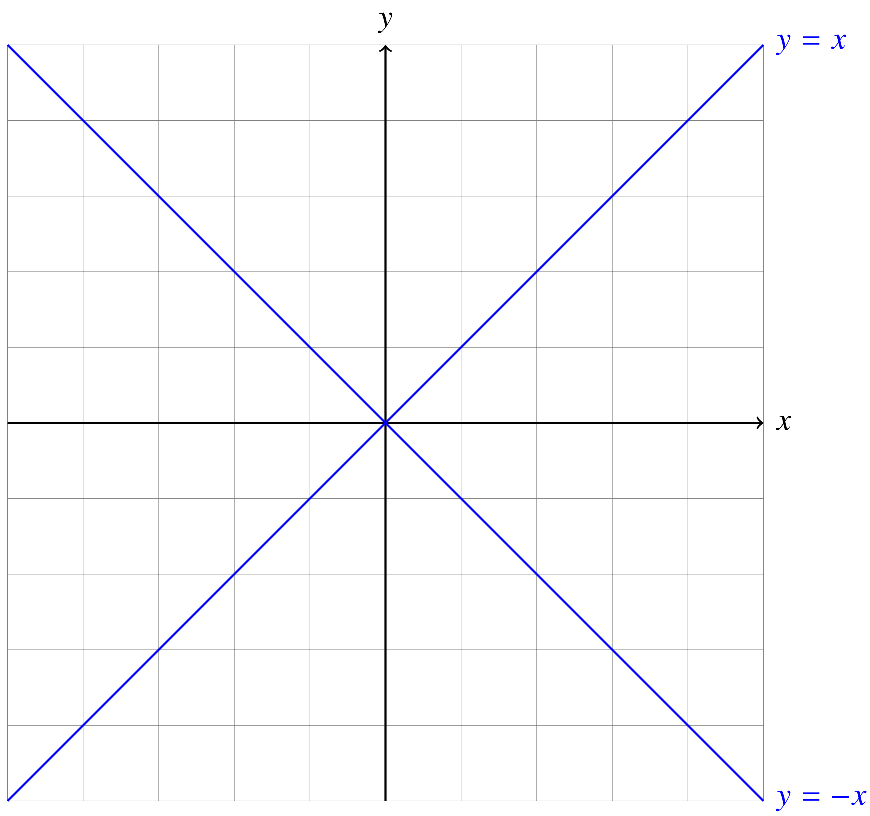

Figure 1: Level curve through the origin

(4) No, because $f(x, y)$ can limit to $\pm\infty$ as x or y limits to $\pm\infty$. Alternatively, we could calculate that f only has one critical point, at $(0, 0)$, and the Hessian determinant is negative there so it is merely a saddle point.

Problem 2.

(1) The graph of f is the plane with equation $2x - y - z + 1 = 0$. It does not go through the origin because the equation is not satisfied for $(x, y, z) = (0, 0, 0)$. The vector $(2, -1, -1)$ is a normal vector to this plane.

(2) It is a straight line, as is always the intersection of two non-parallel planes.

(3) The previous answer shows that any contour curve of f is a straight line in 3-dimensional space. Since the level curves of f are the translations to the xy-plane of the contour curves, they are also straight lines.

(4) The c-level curve of f is the curve with equation $2x - y + 1 - c = 0$ in the xy-plane. It is a straight line. A parallel vector to this line is $\vec{w}_1 = (1, 2)$ and an orthogonal vector is $\vec{w}_2 = (2, -1)$. Note that $\vec{w}_1$ and $\vec{w}_2$ do not depend on c: they work for all level curves.

(5) The c-level curve of f is the straight line with equation $y = 2x + 1 - c = 0$. Here are a few of these lines: (see Figure 2 below)

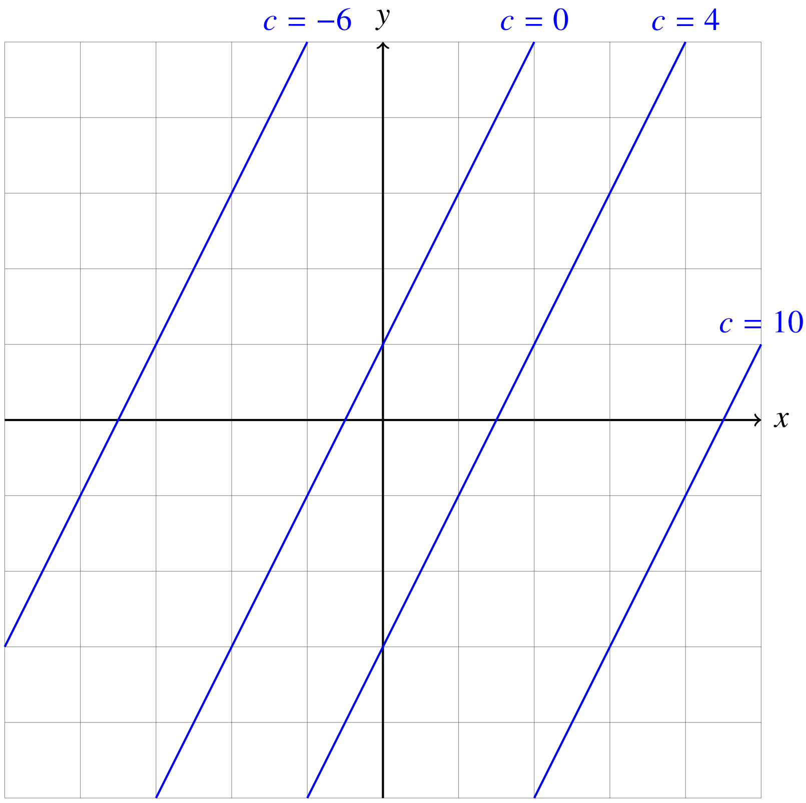

Figure 2: A few level curves of f

(6) $\vec{\nabla} f(x, y) = (2, -1)$. It is parallel to $\vec{w}_2$ (actually they are equal). This is expected because the gradient is always orthogonal to the level curves.

(7) $D_{\vec{w}_1} f(x, y) = \vec{\nabla} f(x, y) \cdot \vec{w}_1 = (2, -1) \cdot (1, 2) = 0$. This is expected because the direction tangent to the level curves is the direction of zero change.

Problem 3.

(1) f is defined on $\mathbb{R}^2$. The gradient of f is $\vec{\nabla} f(x,y) = (6x^2+6y, 6x-6y)$. The coordinates (x,y) of a critical point must satisfy:

$$\begin{cases} 6x^2+6y=0 \\ 6x-6y=0 \end{cases}$$

This is equivalent to:

$$\begin{cases} x^2+x=0 \\ y=x \end{cases}$$

and:

$$\{\; x=0 \text{ or } x=-1 \; y=x$$

Therefore there are two critical points: $P_1(0,0)$ and $P_2(-1,-1)$.
In order to study the nature of these critical points, we do the second derivative test. First we compute the second partial derivatives:

$$r = \frac{\partial^2 f}{\partial x^2}(x,y) = 12x$$
$$s = \frac{\partial^2 f}{\partial y^2}(x,y) = -6$$
$$t = \frac{\partial^2 f}{\partial x \partial y}(x,y) = 6$$

At the point $P_1(0,0)$, we have $rs-t^2=-36<0$, so it is a saddle point. At the point $P_2(-1,-1)$, we have $rs-t^2=36>0$ and $r=-12<0$, so it is a local maximum.

(2) $f(-1,-1)=3$ and $f(1,0)=4$.

(3) No. If f had a global minimum, it would be a local minimum, but f has no local minima. If f had a global maximum, it would be a local maximum. The only local maximum of f is at $(-1,-1)$, but it is not a global maximum since $f(1,0)>f(-1,-1)$.